Math 225 Assignment #10
Due Apr. 7, 2014

(1) Show that the distance from a point \((x_0, y_0, z_0)\) to the plane \(\{ax + by + cz + d = 0\}\) in \(\mathbb{R}^3\) is

\[
\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.
\]

(2) Orthogonally diagonalize the matrix

\[
\begin{bmatrix}
 1 & 1 & 1 \\
 1 & 1 & 1 \\
 1 & 1 & 1
\end{bmatrix}.
\]

(3) Let

\[
A = \begin{bmatrix}
 3 & 1 \\
 4 & 0 \\
 0 & 1
\end{bmatrix}.
\]

(a) Find the QR factorization of \(A\).
(b) Find a vector \(\hat{x} \in \mathbb{R}^2\) minimizing \(||b - Ax||\) for

\[
b = \begin{bmatrix}
 1 \\
 1 \\
 1
\end{bmatrix}.
\]

(4) Which of the following statements are true and which are false? Justify your answer.

(a) The product of two symmetric matrices is also symmetric.
(b) The product of two orthogonal matrices is also orthogonal.
(c) Let \(V\) and \(W\) be two subspaces of \(\mathbb{R}^n\). If \(V \subset W\), then

\[
||\text{proj}_V u|| \leq ||\text{proj}_W u||.
\]
(d) Two symmetric matrices with the same characteristic polynomial must be similar.

(5) Let \(T : \mathbb{R}^n \rightarrow \mathbb{R}^n\) be a linear transformation satisfying

\[
||T(v)|| = ||v||
\]

for all \(v \in \mathbb{R}^n\). Show that

(a) \(\langle T(v_1), T(v_2) \rangle = \langle v_1, v_2 \rangle\) for all \(v_1\) and \(v_2 \in \mathbb{R}^n\).
(b) \([T]_{B,B}\) is an orthogonal matrix, where \(B\) is the standard basis of \(\mathbb{R}^n\).
(6) Apply Gram-Schmidt to find an orthogonal basis for each of the following subspaces of \( \mathbb{R}^4 \):
(a) \( W_1 = \{ x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \} \);
(b) \( W_2 = \text{Span}\{ (1, 0, 1, 0), (0, 1, 1, 1) \} \);
(c) \( W_3 = \{ x_1 + x_2 = 2x_1 + x_3 - x_4 = 0 \} \).

(7) For each of the following quadratic forms, write it in the form of \( x^T A x \) for a symmetric matrix \( A \) and diagonalize it:
(a) \( x_1^2 + x_1 x_2 + x_2^2 \);
(b) \( x_1^2 + x_2^2 - x_3^2 + x_1 x_2 \).

(8) Let \( v \in \mathbb{R}^n \) be a column vector of length 1. Show that \( I - 2vv^T \) is a symmetric orthogonal matrix.