(1) No books, notes or calculators are allowed.
(2) Show your work in details.

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(1) (60 pts) Which of the following statements are true and which are false? Justify your answer.

(a) (10 pts) A matrix $A$ is orthogonal if and only if $A^T$ is.

(b) (10 pts) Let $T : V \to W$ be a linear transformation between two vectors space $V$ and $W$. If $v_1, v_2, \ldots, v_n$ are linearly independent, then $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent.
(c) (10 pts) Let $T_1 : \mathbb{R}^6 \to \mathbb{R}^5$ and $T_2 : \mathbb{R}^5 \to \mathbb{R}^6$ be two linear transformations. Then $T_2 \circ T_1$ cannot be onto.

(d) (10 pts) Two square matrices with the same characteristic polynomial must be similar.
(e) (10 pts) A symmetric matrix with characteristic polynomial 
\((x - 1)^n\) must be the \(n \times n\) identity matrix.

(f) (10 pts) For all linear transformations \(T : V \to V\),
\[K(T^2) \subset K(T).\]
(2) (20 pts) For each of the following quadratic forms, write it in the form of $\mathbf{x}^T A \mathbf{x}$ for a symmetric matrix $A$:

(a) (10 pts) $x_1^2 + 2x_1x_2 + x_2x_3 + x_3^2 + 4x_3x_1$

(b) (10 pts) $x_1^2 + 2x_2^2 - 3x_3^2 - 4x_4^2 - x_1x_4 + 2x_2x_3$
(3) (20 pts) Let $A$ be a $3 \times 3$ matrix whose characteristic polynomial is $x^3 - x$.

(a) (5 pts) Show that $A$ is singular.

(b) (15 pts) Find the characteristic polynomial and eigenvalues of $A + A^2$. 
(4) (20 pts) Let \( \{a_n : n = 0, 1, 2, \ldots\} \) be a sequence of numbers given by

\[
a_n = 2a_{n-1} - 1
\]

for \( n \geq 1 \) and \( a_0 = 2 \). Find a formula for \( a_n \).
(5) (35 pts) Let $M_{m \times n}(\mathbb{R})$ be the vector space of $m \times n$ real matrices and $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ be the map given by

$$T(A) = A - A^T$$

(a) (5 pts) Show that $T$ is a linear transformation.
(b) (10 pts) Find the kernel, range and the rank of $T$. Is $T$ 1-1? Is $T$ onto? Justify your answer.
(c) (10 pts) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$ and find a basis $B$ of $M_{2\times 2}(\mathbb{R})$ such that $[T]_{B,B}$ is diagonal if such $B$ exists.
(d) (10 pts) Find

\[ T^{2014} \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}. \]
(6) (25 pts) Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^4$ given by

$W_1 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 = x_2 + x_3 - x_4 = 0\}$

and

$W_2 = \{(t, 0, t, 0) : t \in \mathbb{R}\}$.

(a) (15 pts) Find the projection of $v = (1, 1, 1, 0)$ onto $W_1 + W_2$. 
(b) (10 pts) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation

$$T(u) = \text{proj}_{W_1 + W_2} u.$$ 

Find the kernel, range and rank of $T$. 
(7) (20 pts) Let $V$ be a vector space of finite dimension and let $T : V \to V$ be a linear transformation satisfying

$$T^2 - 5T + 6I = 0.$$ 

Show that $T$ is diagonalizable.