Matrix Representation of Linear Transformation under Change of Basis

Characteristic Polynomial, Eigenvalues and Eigenvectors of a Linear Endomorphism

Linear Algebra II Lecture 20

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Outline

1. Matrix Representation of Linear Transformation under Change of Basis

2. Characteristic Polynomial, Eigenvalues and Eigenvectors of a Linear Endomorphism
Let $T : V \rightarrow W$ be a linear transformation. Fixing ordered bases $B$ of $V$ and $C$ of $W$, $T$ is represented by a matrix $[T]_{B,C}$ such that

$$[T(v)]_C = [T]_{B,C}[v]_B$$

for all $v \in V$.

Question. Let $B'$ be another ordered basis of $V$ and $C'$ be another ordered basis of $W$. What is the relation between $[T]_{B,C}$ and $[T]_{B',C'}$?
Matrix Representation

Since

\[
\begin{align*}
[T(v)]_C &= [T]_{B,C} [v]_B \\
[T(v)]_{C'} &= P_{C \rightarrow C'} [T(v)]_C \\
[T(v)]_{C'} &= [T]_{B',C'} [v]_{B'} \\
[v]_{B'} &= P_{B \rightarrow B'} [v]_B
\end{align*}
\]

we conclude

\[
P_{C \rightarrow C'} [T]_{B,C} [v]_B = [T]_{B',C'} P_{B \rightarrow B'} [v]_B \quad \text{for all } v \in V
\]
For every $m \times n$ matrix $A$, there are invertible matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where $I_k$ is the $k \times k$ identity matrix and $k = \text{rank}(A)$. For every linear transformation $T : V \to W$, there are bases $B'$ of $V$ and $C'$ of $W$ such that

$$[T]_{B',C'} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where $k = \text{rank}(T)$. It suffices to let $[T]_{B,C} = A$, $P_{C \to C'} = P$ and $P_{B' \to B} = Q$. 

Xi Chen
Linear Algebra II Lecture 20
Let $k$ be the rank of $T : V \to W$. Then
\[
\dim K(T) = \dim V - \text{rank}(T) = n - k
\]
assuming $\dim V = n$ and $\dim W = m$.

We choose a basis
\[
B' = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}
\]
of $V$ such that $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ is a basis of $K(T)$ and choose a basis
\[
C' = \left\{ \begin{array}{c}
T(v_1) \\
T(v_2) \\
\vdots \\
T(v_k) \\
w_{k+1} \\
\vdots \\
w_m
\end{array} \right\}
\]
of $W$ (so that $w_1, w_2, \ldots, w_k$ is a basis of $R(T)$).
Since

\[ T(v_1) = w_1 \]
\[ T(v_2) = 0w_1 + w_2 \]
\[ \vdots = \cdots \]
\[ T(v_k) = 0w_1 + 0w_2 + \ldots + 0w_{k-1} + w_k \]
\[ T(v_{k+1}) = 0 \]
\[ \vdots = \vdots \]
\[ T(v_n) = 0 \]

we obtain

\[ [T]_{B',C'} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} \]
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y, z) = (x + y, y + z).$$

The kernel of $T$ is $K(T) = \text{Span}\{(1, -1, 1)\}$. We choose $B = \{(1, 0, 0), (0, 1, 0), (1, -1, 1)\}$ and $C = \{T(1, 0, 0), T(0, 1, 0)\} = \{(1, 0), (1, 1)\}$. Then

$$[T]_{B,C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$
Linear Endomorphism

We call a linear transformation \( T : V \rightarrow V \) from a vector space \( V \) to itself a \textit{linear endomorphism} or simply \textit{endomorphism}.

Given two ordered bases \( B \) and \( B' \) of \( V \), \([T]_{B,B}\) and \([T]_{B',B'}\) satisfy

\[
[T]_{B',B'} = P_{B \rightarrow B'} [T]_{B,B} P_{B \rightarrow B'}^{-1} = P [T]_{B,B} P^{-1}.
\]

Therefore, \([T]_{B,B}\) and \([T]_{B',B'}\) are similar to each other.

For example, let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by

\( T(x, y) = (x + y, x - y) \), \( B = \{ e_1, e_2 \} \) and \( B' = \{ (1, 2), (2, 3) \} \).

Then

\[
\begin{bmatrix}
-11 & -17 \\
7 & 11
\end{bmatrix} = \begin{bmatrix}
-3 & 2 \\
2 & -1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix}
\]
Characteristic Polynomial, Eigenvalue and Eigenvector

Let $T : V \rightarrow V$ be a linear endomorphism. We call $\lambda$ an \textit{eigenvalue} of $T$ and $v \neq 0$ an \textit{eigenvector} of $T$ corresponding to $\lambda$ if $T(v) = \lambda v$.

If $\dim V = n < \infty$, the eigenvalues and eigenvectors of $T$ are the same as those of $[T]_{B,B}$ for all ordered bases $B$ of $V$.

We call

$$\det(x I - [T]_{B,B}) = x^n + a_1 x^{n-1} + \ldots + a_n$$

the \textit{characteristic polynomial} of $T$, which is independent of the choice of $B$. Note that $a_n = \det(-[T]_{B,B}) = (-1)^n \det([T]_{B,B})$ and $a_1 = -\text{Tr}([T]_{B,B})$ is the trace of $[T]_{B,B}$. 