Outline

1. Matrix Representation of Linear Transformation under Change of Basis

2. Characteristic Polynomial, Eigenvalues and Eigenvectors of a Linear Endomorphism
Let $T : V \to W$ be a linear transformation. Fixing ordered bases $B$ of $V$ and $C$ of $W$, $T$ is represented by a matrix $[T]_{B,C}$ such that

$$[T(v)]_C = [T]_{B,C}[v]_B$$

for all $v \in V$.

Question. Let $B'$ be another ordered basis of $V$ and $C'$ be another ordered basis of $W$. What is the relation between $[T]_{B,C}$ and $[T]_{B',C'}$?
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Question. Let $B'$ be another ordered basis of $V$ and $C'$ be another ordered basis of $W$. What is the relation between $[T]_{B,C}$ and $[T]_{B',C'}$?
Matrix Representation

Since

\[
\begin{align*}
[T(\mathbf{v})]_C &= [T]_{B,C}[\mathbf{v}]_B \\
[T(\mathbf{v})]_{C'} &= P_{C \rightarrow C'}[T(\mathbf{v})]_C \\
[T(\mathbf{v})]_{C'} &= [T]_{B',C'}[\mathbf{v}]_{B'} \\
[\mathbf{v}]_{B'} &= P_{B \rightarrow B'}[\mathbf{v}]_B
\end{align*}
\]

we conclude

\[
P_{C \rightarrow C'}[T]_{B,C}[\mathbf{v}]_B = [T]_{B',C'}P_{B \rightarrow B'}[\mathbf{v}]_B \quad \text{for all } \mathbf{v} \in V
\]
Since

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\begin{align*}
[T(v)]_C &= [T]_{B,C}[v]_B \\
[T(v)]_{C'} &= P_{C \rightarrow C'}[T(v)]_C \\
[T(v)]_{C'} &= [T]_{B',C'}[v]_{B'} \\
[v]_{B'} &= P_{B \rightarrow B'}[v]_B
\end{align*}
\]

we conclude

\[
P_{C \rightarrow C'}[T]_{B,C}[v]_B = [T]_{B',C'}P_{B \rightarrow B'}[v]_B \quad \text{for all } v \in V
\]

\[
P_{C \rightarrow C'}[T]_{B,C} = [T]_{B',C'}P_{B \rightarrow B'} \iff [T]_{B',C'} = P_{C \rightarrow C'}[T]_{B,C}P_{B \rightarrow B'}^{-1}
\]
Matrix Representation

For every \( m \times n \) matrix \( A \), there are invertible matrices \( P \) and \( Q \) such that

\[
PAQ = \begin{bmatrix}
I_k & 0 \\
0 & 0 \\
\end{bmatrix}
\]

where \( I_k \) is the \( k \times k \) identity matrix and \( k = \text{rank}(A) \).

For every linear transformation \( T : V \to W \), there are bases \( B' \) of \( V \) and \( C' \) of \( W \) such that

\[
[T]_{B',C'} = \begin{bmatrix}
I_k & 0 \\
0 & 0 \\
\end{bmatrix}
\]

where \( k = \text{rank}(T) \). It suffices to let \([T]_{B,C} = A\), \( P_{C \to C'} = P \) and \( P_{B' \to B} = Q \).
Matrix Representation

For every $m \times n$ matrix $A$, there are invertible matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where $I_k$ is the $k \times k$ identity matrix and $k = \text{rank}(A)$. For every linear transformation $T : V \to W$, there are bases $B'$ of $V$ and $C'$ of $W$ such that

$$[T]_{B',C'} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where $k = \text{rank}(T)$. It suffices to let $[T]_{B,C} = A$, $P_{C \to C'} = P$ and $P_{B' \to B} = Q$. 
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Linear Algebra II Lecture 20
Matrix Representation

Let $k$ be the rank of $T : V \to W$. Then
\[ \dim K(T) = \dim V - \text{rank}(T) = n - k \]
assuming $\dim V = n$ and $\dim W = m$.

We choose a basis
\[ B' = \{ v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n \} \]
of $V$ such that $\{ v_{k+1}, v_{k+2}, \ldots, v_n \}$ is a basis of $K(T)$ and choose a basis
\[ C' = \{ \underbrace{w_1}_{T(v_1)}, \underbrace{w_2}_{T(v_2)}, \ldots, \underbrace{w_k}_{T(v_k)}, w_{k+1}, \ldots, w_m \} \]
of $W$ (so that $w_1, w_2, \ldots, w_k$ is a basis of $R(T)$).
Matrix Representation

Let $k$ be the rank of $T : V \to W$. Then $\dim K(T) = \dim V - \text{rank}(T) = n - k$ assuming $\dim V = n$ and $\dim W = m$.

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Let $k$ be the rank of $T : V \rightarrow W$. Then
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We choose a basis
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of $W$ (so that $w_1, w_2, \ldots, w_k$ is a basis of $R(T)$).
Matrix Representation

Since

\[ T(v_1) = w_1 \]
\[ T(v_2) = 0w_1 + w_2 \]
\[ \vdots = \cdots \]
\[ T(v_k) = 0w_1 + 0w_2 + \ldots + 0w_{k-1} + w_k \]
\[ T(v_{k+1}) = 0 \]
\[ \vdots = \vdots \]
\[ T(v_n) = 0 \]

we obtain

\[ [T]_{B',C'} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} \]
Matrix Representation

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we obtain

\[ [T]_{B',C'} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} \]
Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T(x, y, z) = (x + y, y + z).$$

The kernel of $T$ is $K(T) = \text{Span}\{(1, -1, 1)\}$. We choose $B = \{(1, 0, 0), (0, 1, 0), (1, -1, 1)\}$ and $C = \{T(1, 0, 0), T(0, 1, 0)\} = \{(1, 0), (1, 1)\}$. Then

$$[T]_{B,C} = \begin{bmatrix}
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Linear Endomorphism

We call a linear transformation $T : V \rightarrow V$ from a vector space $V$ to itself a **linear endomorphism** or simply **endomorphism**. Given two ordered bases $B$ and $B'$ of $V$, $[T]_{B,B}$ and $[T]_{B',B'}$ satisfy

$$[T]_{B',B'} = P_{B \rightarrow B'} [T]_{B,B} P_{B \rightarrow B'}^{-1} = P[T]_{B,B} P^{-1}.$$

Therefore, $[T]_{B,B}$ and $[T]_{B',B'}$ are similar to each other. For example, let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $T(x, y) = (x + y, x - y)$, $B = \{e_1, e_2\}$ and $B' = \{(1, 2), (2, 3)\}$. Then

$$\begin{bmatrix} -11 & -17 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

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Linear Algebra II Lecture 20
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$$[T]_{B',B'} = \begin{bmatrix} -11 & -17 \\ 7 & 11 \end{bmatrix}, \quad P^{-1}_{B'\rightarrow B} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} [T]_{B,B} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$
Linear Endomorphism

We call a linear transformation \( T : V \to V \) from a vector space \( V \) to itself a *linear endomorphism* or simply *endomorphism*. Given two ordered bases \( B \) and \( B' \) of \( V \), \([T]_{B,B} \) and \([T]_{B',B'} \) satisfy

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= P_{B \to B'}^{-1} [T]_{B,B} P_{B \to B'}^{-1}.
\]
Let $T : V \rightarrow V$ be a linear endomorphism. We call $\lambda$ an eigenvalue of $T$ and $v \neq 0$ an eigenvector of $T$ corresponding to $\lambda$ if $T(v) = \lambda v$.

If $\text{dim } V = n < \infty$, the eigenvalues and eigenvectors of $T$ are the same as those of $[T]_{B,B}$ for all ordered bases $B$ of $V$.

We call

$$\det(xI - [T]_{B,B}) = x^n + a_1 x^{n-1} + \ldots + a_n$$

the characteristic polynomial of $T$, which is independent of the choice of $B$. Note that $a_n = \det(-[T]_{B,B}) = (-1)^n \det([T]_{B,B})$ and $a_1 = -\text{Tr}([T]_{B,B})$ is the trace of $[T]_{B,B}$.
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Let $T : V \to V$ be a linear endomorphism. We call $\lambda$ an \textit{eigenvalue} of $T$ and $v \neq 0$ an \textit{eigenvector} of $T$ corresponding to $\lambda$ if $T(v) = \lambda v$.

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