Change of Basis
Matrix Representation of Linear Transformation under Change of Basis

Linear Algebra II Lecture 19

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Outline

1. Change of Basis

2. Matrix Representation of Linear Transformation under Change of Basis
Let $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_n\}$ be two ordered bases of $V$. The \textit{change-of-coordinates} matrices $P_{B \rightarrow C}$ and $P_{C \rightarrow B}$ are the matrices such that

$$[v]_C = P_{B \rightarrow C}[v]_B \text{ and } [v]_B = P_{C \rightarrow B}[v]_C$$

for all $v \in V$.

More specifically,

$$P_{B \rightarrow C} = \begin{bmatrix} [v_1]_C & [v_2]_C & \cdots & [v_n]_C \end{bmatrix}$$

$$P_{C \rightarrow B} = \begin{bmatrix} [w_1]_B & [w_2]_B & \cdots & [w_n]_B \end{bmatrix}$$
Basic Properties of $P_{B \to C}$

- $P_{B \to C}$ is invertible and
  
  \[ P_{B \to C} = (P_{C \to B})^{-1} \]

- Let $B$, $C$ and $D$ be three bases of $V$. Then

  \[ P_{B \to D} = P_{C \to D} P_{B \to C} \]

  since

  \[ [v]_D = P_{C \to D} [v]_C = P_{C \to D} (P_{B \to C} [v]_B) = (P_{C \to D} P_{B \to C}) [v]_B. \]
Rotate $xy$-coordinates counter clockwise by an angle $\theta$. What are the new coordinates $(x', y')$ of a point $P = (x, y)$?
Two-dimensional Rotation

Let $B = \{e_1, e_2\}$ be the standard basis. Rotating $B$ counter clockwise by $\theta$, we obtain

$$B' = \{v_1, v_2\} = \{(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)\}.$$

The change-of-coordinates matrices between $B$ and $B'$ are

$$P_{B' \rightarrow B} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad P_{B \rightarrow B'} = P_{B' \rightarrow B}^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

Note that $P_{B \rightarrow B'} = P_{B' \rightarrow B}^T$. Recall that such matrices are called orthogonal.
Two-dimensional Rotation

For a point \( \mathbf{v} = (x, y) \) under \( B \),

\[
[\mathbf{v}]_{B'} = \begin{bmatrix} x' \\ y' \end{bmatrix} = P_{B \rightarrow B'}[\mathbf{v}]_B = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

\[
= \begin{bmatrix} x \cos \theta + y \sin \theta \\ -x \sin \theta + y \cos \theta \end{bmatrix}.
\]

Coordinate change formula of a two-dimensional rotation:

\[
\begin{cases}
  x' = x \cos \theta + y \sin \theta \\
y' = -x \sin \theta + y \cos \theta
\end{cases}
\]

For example, rotating \( xy \)-coordinates by \( \pi/3 \), \( P = (1, 1) \) has new coordinates

\[
(x', y') = \left( \cos \frac{\pi}{3} + \sin \frac{\pi}{3}, -\sin \frac{\pi}{3} + \cos \frac{\pi}{3} \right) = \left( \frac{1}{2} + \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} + \frac{1}{2} \right).
\]
Let $T : V \rightarrow W$ be a linear transformation. Fixing ordered bases $B$ of $V$ and $C$ of $W$, $T$ is represented by a matrix $[T]_{B,C}$ such that

$$[T(v)]_C = [T]_{B,C}[v]_B$$

for all $v \in V$.

Question. Let $B'$ be another ordered basis of $V$ and $C'$ be another ordered basis of $W$. What is the relation between $[T]_{B,C}$ and $[T]_{B',C'}$?
Matrix Representation

Since

\[
\begin{cases}
[T(\mathbf{v})]_C = [T]_{B,C}[\mathbf{v}]_B \\
[T(\mathbf{v})]_{C'} = P_{C \rightarrow C'} [T(\mathbf{v})]_C \\
[T(\mathbf{v})]_{C'} = [T]_{B', C'} [\mathbf{v}]_{B'} \\
[\mathbf{v}]_{B'} = P_{B \rightarrow B'} [\mathbf{v}]_B
\end{cases}
\]

we conclude

\[
P_{C \rightarrow C'} [T]_{B,C}[\mathbf{v}]_B = [T]_{B', C'} P_{B \rightarrow B'} [\mathbf{v}]_B \text{ for all } \mathbf{v} \in V
\]