Linear Algebra II Lecture 17

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1. Consequences of Rank Theorem
Let $T : V \to W$ be a linear transformation between two vector spaces of dimensions $\dim V = n$ and $\dim W = m$ and let $A = [T]$ be the matrix representing $T$. Then

\[ T \text{ 1-1 } \iff K(T) = \{0\} \iff \text{Nul}(A) = \{0\}. \]

\[ T \text{ onto } \iff R(T) = W \iff \text{rank}(T) = \dim W \iff \text{rank}(A) = m. \]

\[ T \text{ bijective } \iff T \text{ 1-1 and onto } \iff K(T) = \{0\}, \text{rank}(T) = m \]

\[ A \text{ invertible } \iff \text{Nul}(A) = \{0\}, \text{rank}(A) = m. \]
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\end{align*}

$T$ bijective $\iff T$ 1-1 and onto $\iff K(T) = \{0\}$, $\text{rank}(T) = m$

$A$ invertible $\iff$ Nul$(A) = \{0\}$, $\text{rank}(A) = m$
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$A$ invertible $\iff \text{Nul}(A) = \{0\}$, $\text{rank}(A) = m$
Corollary

Let $A$ be an $m \times n$ matrix. Then

- $\text{rank}(A) + \text{dim Nul}(A) = n$.
- The solution set of $Ax = 0$ is a subspace of $\mathbb{R}^n$ of dimension $n - \text{rank}(A)$. In particular, if $m < n$, $Ax = 0$ has infinitely many solutions.
- $Ax = b$ has a solution if and only if $b \in \text{Col}(A)$, or equivalently, $\text{rank}(A) = \text{rank} \begin{bmatrix} A & b \end{bmatrix}$. If $x_0$ is a solution of $Ax = b$, then
  \[
  \{ x : Ax = b \} = x_0 + \text{Nul}(A) = \{ x_0 + v : v \in \text{Nul}(A) \}.
  \]
  In particular, if $m < n$, $Ax = b$ has either no solutions or infinitely many solutions.
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In particular, if $m < n$, $Ax = b$ has either no solutions or infinitely many solutions.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $T(x) = Ax$. Then

$$[T] = A, \dim K(T) = \dim \text{Nul}(A),$$
$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$

$Ax = b$ has a solution if and only if $T(x) = b$ for some $x$, i.e.,

$$b \in R(T) \iff b \in \text{Col}(A) \iff \text{Col}(A) = \text{Col} \begin{bmatrix} A & b \end{bmatrix} \iff \text{rank}(A) = \text{rank} \begin{bmatrix} A & b \end{bmatrix}. $$

Finally, if $x_0$ and $x_1$ are two solutions of $Ax = b$. Then

$$T(x_1 - x_0) = T(x_1) - T(x_0) = b - b = 0,$$
$$x_1 - x_0 \in K(T) = \text{Nul}(A) \Rightarrow x_1 \in x_0 + \text{Nul}(A).$$
Proof.

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the linear transformation given by $T(x) = Ax$. Then

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b \in R(T) \iff b \in \text{Col}(A)
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x_1 - x_0 \in K(T) = \text{Nul}(A) \quad \Rightarrow \quad x_1 \in x_0 + \text{Nul}(A).
\]
**Corollary**

Let $A$ be an $n \times n$ matrix. Then the following are equivalent:

- $A$ is nonsingular ($\det A \neq 0$).
- $A$ is invertible ($A$ has an inverse $A^{-1}$ satisfying $AA^{-1} = A^{-1}A = I$).
- $\text{Nul}(A) = \{0\}$.
- $Ax = 0$ has only one solution.
- $\text{rank}(A) = n$.
- $Ax = b$ has a solution for all $b \in \mathbb{R}^n$. 
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- $A\mathbf{x} = 0$ has only one solution.
- $\text{rank}(A) = n$.
- $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$. 
Corollary

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Proof.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by 
$T(x) = Ax$. Then $[T] = A$ and

\[
\text{det } A \neq 0 \iff A \text{ invertible } \iff T \text{ bijective}
\]

\[
\text{Nul}(A) = \{0\} \iff T \text{ injective}
\]

\[
\text{rank}(A) = n \iff T \text{ onto}
\]

where CR = Cramer’s Rule and RT = Rank Theorem.
Let $T : V \to W$ and $S : U \to V$ be two linear transformations between vector spaces $U$, $V$ and $W$ of finite dimensions. Then

- $\text{rank}(T \circ S) \leq \min(\text{rank}(T), \text{rank}(S))$.
- $\text{rank}(T \circ S) = \text{rank}(T)$ if $S$ is onto.
- $\text{rank}(T \circ S) = \text{rank}(S)$ if $T$ is 1-1.
- In summary, $\text{rank}(T \circ S) = \min(\text{rank}(T), \text{rank}(S))$ if either $\text{rank}(T) = \dim V$ or $\text{rank}(S) = \dim V$.

For example,

$$\text{rank}(T_1 \circ T_2 \circ \ldots \circ T_n) \leq \min(\text{rank}(T_1), \text{rank}(T_2), \ldots, \text{rank}(T_n))$$

and the equality holds if all of $T_i$, except one, are bijective.
Theorem

Let $T : V \rightarrow W$ and $S : U \rightarrow V$ be two linear transformations between vector spaces $U$, $V$ and $W$ of finite dimensions. Then

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Let $T : V \rightarrow W$ and $S : U \rightarrow V$ be two linear transformations between vector spaces $U$, $V$ and $W$ of finite dimensions. Then

- $\text{rank}(T \circ S) \leq \min(\text{rank}(T), \text{rank}(S))$.
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and the equality holds if all of $T_i$, except one, are bijective.
Consequences of Rank Theorem

Ranks of Compositions of Linear Transformations

**Theorem**

Let $T : V \rightarrow W$ and $S : U \rightarrow V$ be two linear transformations between vector spaces $U$, $V$ and $W$ of finite dimensions. Then

- $\text{rank}(T \circ S) \leq \min(\text{rank}(T), \text{rank}(S))$.
- $\text{rank}(T \circ S) = \text{rank}(T)$ if $S$ is onto.
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In summary, $\text{rank}(T \circ S) = \min(\text{rank}(T), \text{rank}(S))$ if either $\text{rank}(T) = \dim V$ or $\text{rank}(S) = \dim V$.

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$$\text{rank}(T_1 \circ T_2 \circ \ldots \circ T_n) \leq \min(\text{rank}(T_1), \text{rank}(T_2), \ldots, \text{rank}(T_n))$$

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Theorem
Let \( T : V \rightarrow W \) and \( S : U \rightarrow V \) be two linear transformations between vector spaces \( U, V \) and \( W \) of finite dimensions. Then

1. \( \text{rank}(T \circ S) \leq \min(\text{rank}(T), \text{rank}(S)) \).
2. \( \text{rank}(T \circ S) = \text{rank}(T) \) if \( S \) is onto.
3. \( \text{rank}(T \circ S) = \text{rank}(S) \) if \( T \) is 1-1.
4. In summary, \( \text{rank}(T \circ S) = \min(\text{rank}(T), \text{rank}(S)) \) if either \( \text{rank}(T) = \dim V \) or \( \text{rank}(S) = \dim V \).

For example,

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\text{rank}(T_1 \circ T_2 \circ \ldots \circ T_n) \leq \min(\text{rank}(T_1), \text{rank}(T_2), \ldots, \text{rank}(T_n))
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and the equality holds if all of \( T_i \), except one, are bijective.
Proof.

Note that

\[ \text{rank}(T \circ S) \leq \min(\text{rank}(T), \text{rank}(S)) \iff \begin{cases} 
\text{rank}(T \circ S) \leq \text{rank}(T) \\
\text{rank}(T \circ S) \leq \text{rank}(S) 
\end{cases} \]

Since \( S(U) \subset V, T(S(U)) \subset T(V) \), i.e., \( R(T \circ S) \subset R(T) \). So

\[ \text{rank}(T \circ S) \leq \text{rank}(T). \]

By Rank Theorem, \( \dim T(S(U)) \leq \dim S(U) \). Therefore,

\[ \text{rank}(T \circ S) \leq \text{rank}(S). \]
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By Rank Theorem, \( \dim T(S(U)) \leq \dim S(U) \). Therefore,

\[ \text{rank}(T \circ S) \leq \text{rank}(S). \]
Proof.

Suppose that $S$ is surjective. Then $S(U) = V$ and hence $T(S(U)) = T(V)$, i.e., $R(T \circ S) = R(T)$. So

$$\text{rank}(T \circ S) = \text{rank}(T).$$

Suppose that $T$ is injective. By Rank Theorem,

$$\dim T(S(U)) = \dim S(U).$$

Therefore,

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Proof.

Suppose that $S$ is surjective. Then $S(U) = V$ and hence $T(S(U)) = T(V)$, i.e., $R(T \circ S) = R(T)$. So

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Corollary

Let $A$ be an $l \times m$ matrix and $B$ be an $m \times n$ matrix. Then

- $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \leq \min(l, m, n)$.
- $\text{rank}(AB) = \text{rank}(A)$ if $\text{rank}(B) = m$.
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- In summary, $\text{rank}(AB) = \min(\text{rank}(A), \text{rank}(B))$ if either $\text{rank}(A) = m$ or $\text{rank}(B) = m$.

For example,

$$\text{rank} \left( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \right) \leq 1.$$
Consequences of Rank Theorem

Ranks of Products of Matrices

Corollary

Let $A$ be an $l \times m$ matrix and $B$ be an $m \times n$ matrix. Then

- $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \leq \min(l, m, n)$.
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$$\text{rank} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \leq 1.$$
**Corollary**

Let $A$ be an $l \times m$ matrix and $B$ be an $m \times n$ matrix. Then

1. $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \leq \min(l, m, n)$.  
2. $\text{rank}(AB) = \text{rank}(A)$ if $\text{rank}(B) = m$.  
3. $\text{rank}(AB) = \text{rank}(B)$ if $\text{rank}(A) = m$.  
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$$
\text{rank} \left( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \right) \leq 1.
$$
Corollary

Let $A$ be an $m \times n$ matrix. Then

- $\text{rank}(A) = \text{rank}(PAQ)$ for all nonsingular matrices $P$ and $Q$.
- $A$ has rank $k$ if and only if there exist nonsingular matrices $P$ and $Q$ such that
  \[
  PAQ = \begin{bmatrix}
  I_k & 0 \\
  0 & 0
  \end{bmatrix}_{m \times n}
  \]
  where $I_k$ is the $k \times k$ identity matrix. In other words, $A$ can be reduced to the above matrix by a sequence of row and column reductions.
- $A$ has rank $k$ if there exists an $m \times k$ matrix $B$ and a $k \times n$ matrix $C$, both of rank $k$, such that $A = BC$.
- $\text{rank}(A) = \text{rank}(A^T)$, i.e., $\dim \text{Row}(A) = \dim \text{Col}(A)$. 
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Consequences of Rank Theorem

Ranks of Products of Matrices
Corollary

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- $\text{rank}(A) = \text{rank}(A^T)$, i.e., $\text{dim} \text{ Row}(A) = \text{dim} \text{ Col}(A)$. 
Proof of $A = BC$ and $\text{rank}(A) = \text{rank}(A^T)$.

Suppose that $A$ has rank $k$. Then

$$PAQ = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \iff A = P^{-1} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Let

$$B = P^{-1} \begin{bmatrix} I_k \\ 0 \end{bmatrix}_{m \times k} \quad \text{and} \quad C = \begin{bmatrix} I_k & 0 \end{bmatrix}_{k \times n} Q^{-1}.$$  

Then $A = BC$. Obviously, $\text{rank}(B) = \text{rank}(C) = k$.

Since

$$PAQ = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} \quad \Rightarrow \quad Q^T A^T P^T = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}$$

$\text{rank}(A^T) = k = \text{rank}(A)$. 

Xi Chen  
Linear Algebra II Lecture 17
Proof of $A = BC$ and $\text{rank}(A) = \text{rank}(A^T)$.

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$\text{rank}(A^T) = k = \text{rank}(A)$. 
Let 

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \]

Find the rank of \( A \) and invertible matrices \( P \) and \( Q \) such that

\[ PAQ = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}. \]

First we apply row reductions to \([A \quad I_3]\):

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
2 & -1 & -1 & 0 & 0 & 1 & 0 \\
0 & 2 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Examples of Finding the Ranks of Matrices

Let

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \]

Find the rank of \( A \) and invertible matrices \( P \) and \( Q \) such that

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\[
\begin{bmatrix}
\text{1} & 1 & 1 & 1 & 1 & 0 & 0 \\
2 & -1 & -1 & 0 & 0 & 1 & 0 \\
0 & 2 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]
Examples of Finding the Ranks of Matrices

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & -3 & -3 & -2 & -2 & 1 \\
0 & 2 & 1 & 1 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
1 & 3 \\
2 & 3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
1 & 3 \\
2 & 3 \\
0 & -1 \\
-1 & 3 \\
4 & 3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Therefore, \( \text{rank}(A) = 3 \) and
Therefore, rank(A) = 3 and
Then we apply column reductions to
Consequences of Rank Theorem

Examples of Finding the Ranks of Matrices

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{3} \\
0 & 1 & 0 & \frac{1}{3} \\
0 & 0 & 1 & \frac{1}{3}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & 0 \\
-\frac{2}{3} & \frac{1}{3} & 1 \\
\frac{4}{3} & -\frac{2}{3} & -1
\end{bmatrix}_P
\]

Then we apply column reductions to

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{3} \\
0 & 1 & 0 & \frac{1}{3} \\
0 & 0 & 1 & \frac{1}{3} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & -\frac{1}{3} \\
0 & 0 & 1 & -\frac{1}{3} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Therefore,

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{3} \\
0 & 1 & 0 & \frac{1}{3} \\
0 & 0 & 1 & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
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\end{bmatrix}
\]

Xi Chen
Linear Algebra II Lecture 17
Theorem

Let \( A \) be an \( m \times n \) matrix. Then

\[
\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T).
\]

Proof.

By Rank Theorem, it suffices to show that \( \text{Nul}(A) = \text{Nul}(A^T A) \).

Note that we always have \( \text{Nul}(A) \subset \text{Nul}(PA) \) for all matrices \( P \)
and hence \( \text{Nul}(A) \subset \text{Nul}(A^T A) \). It suffices to prove that \( \text{Nul}(A^T A) \subset \text{Nul}(A) \).

Let \( x \in \text{Nul}(A^T A) \). That is, \( (A^T A)x = 0 \) and hence

\[
x^T A^T A x = 0 \Rightarrow (Ax)^T (Ax) = 0 \Rightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0.
\]

Namely, \( x \in \text{Nul}(A) \) and \( \text{Nul}(A^T A) \subset \text{Nul}(A) \).
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Xi Chen

Linear Algebra II Lecture 17
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Let $x \in \text{Nul}(A^T A)$. That is, $(A^T A)x = 0$ and hence

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Namely, $x \in \text{Nul}(A)$ and $\text{Nul}(A^T A) \subset \text{Nul}(A)$. 