Outline

1. Rank of a Linear Transformation

2. Consequences of Rank Theorem
Rank

**Definition**

Let \( T : V \rightarrow W \) be a linear transformation. The *rank* of \( T \) is the dimension of its range \( R(T) \), i.e.,

\[
\text{rank}(T) = \dim R(T).
\]

**Theorem**

Let \( T : V \rightarrow W \) be a linear transformation between two vector spaces of finite dimensions. Then

\[
\text{rank}(T) = \text{rank}[T]_{B,C}
\]

where \([T]_{B,C}\) is the matrix representing \( T \) under bases \( B \) and \( C \).
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where $[T]_{B,C}$ is the matrix representing $T$ under bases $B$ and $C$. 
Rank Theorem

Let $T : V \rightarrow W$ be a linear transformation. If $\dim V < \infty$, then

$$\dim K(T) + \text{rank}(T) = \dim V.$$ 

Here are some remarks:

- $\text{rank}(T) \leq \dim V$; and since $R(T) \subset W$, $\text{rank}(T) \leq \dim W$; therefore,
  $$\text{rank}(T) \leq \min(\dim V, \dim W).$$

For example, a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ has rank at most 3.
Let $T : V \to W$ be a linear transformation. If $\dim V < \infty$, then
\[ \dim K(T) + \text{rank}(T) = \dim V. \]

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Linear Algebra II Lecture 15
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Rank Theorem

- \( T \) cannot be onto if \( \dim V < \dim W \) since
  \[
  \dim R(T) = \text{rank}(T) \leq \dim V < \dim W \Rightarrow R(T) \neq W.
  \]

- \( T \) cannot be 1-1 if \( \dim V > \dim W \) since
  \[
  \dim K(T) = \dim V - \text{rank}(T) \geq \dim V - \dim W > 0
  \Rightarrow K(T) \neq \{0\}.
  \]

- \( \dim R(T) = \dim V \) if and only if \( K(T) = \{0\} \), i.e., \( T \) is 1-1.
- If \( \dim V = \dim W \), \( T \) is 1-1 if and only if \( T \) is onto since
  \[
  T \text{ is 1-1 } \iff K(T) = \{0\} \iff \dim K(T) = 0
  \iff \text{rank}(T) = \dim V = \dim W \iff R(T) = W
  \]
Rank Theorem

- **T cannot be onto if dim V < dim W** since
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- **T cannot be 1-1 if dim V > dim W** since
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- **dim R(T) = dim V if and only if K(T) = \{0\}, i.e., T is 1-1.**
- **If dim V = dim W, T is 1-1 if and only if T is onto since**
  \[ T \text{ is 1-1} \iff K(T) = \{0\} \iff \text{dim } K(T) = 0 \]
  \[ \iff \text{rank}(T) = \text{dim } V = \text{dim } W \iff R(T) = W. \]
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  \[ T \text{ is 1-1 } \iff K(T) = \{0\} \iff \dim K(T) = 0 \]
  \[ \iff \text{rank}(T) = \dim V = \dim W \iff R(T) = W \]
Let $\dim K(T) = k$ and $\text{rank}(T) = r$. There is a basis
\[ \{v_1, v_2, ..., v_k\} \] for $K(T)$ and a basis \[ \{w_1, w_2, ..., w_r\} \] for $R(T)$. Since $w_1, w_2, ..., w_r \in R(T)$, there exist $u_1, u_2, ..., u_r \in V$ such that
\[ T(u_1) = w_1, T(u_2) = w_2, ..., T(u_r) = w_r. \]
It suffices to show that \[ \{v_1, v_2, ..., v_k, u_1, u_2, ..., u_r\} \] is a basis of $V$. That is,
- \[ V = \text{Span}\{u_1, u_2, ..., u_r, v_1, v_2, ..., v_k\}; \]
- $u_1, u_2, ..., u_r, v_1, v_2, ..., v_k$ are linearly independent.
Let $\dim K(T) = k$ and $\text{rank}(T) = r$. There is a basis 
$\{v_1, v_2, \ldots, v_k\}$ for $K(T)$ and a basis $\{w_1, w_2, \ldots, w_r\}$ for $R(T)$. Since $w_1, w_2, \ldots, w_r \in R(T)$, there exist $u_1, u_2, \ldots, u_r \in V$ such that

$$T(u_1) = w_1, \ T(u_2) = w_2, \ldots, \ T(u_r) = w_r.$$

It suffices to show that $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_r\}$ is a basis of $V$. That is,

- $V = \text{Span}\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_k\}$;
- $u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_k$ are linearly independent.
Let \( \text{dim } K(T) = k \) and \( \text{rank}(T) = r \). There is a basis \( \{v_1, v_2, ..., v_k\} \) for \( K(T) \) and a basis \( \{w_1, w_2, ..., w_r\} \) for \( R(T) \). Since \( w_1, w_2, ..., w_r \in R(T) \), there exist \( u_1, u_2, ..., u_r \in V \) such that

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T(u_1) = w_1, \quad T(u_2) = w_2, \quad ..., \quad T(u_r) = w_r.
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It suffices to show that \( \{v_1, v_2, ..., v_k, u_1, u_2, ..., u_r\} \) is a basis of \( V \). That is,

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Proof of $V = \text{Span}\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_k\}$

For every $v \in V$, $T(v) \in R(T)$ and hence

$$T(v) = a_1w_1 + a_2w_2 + \ldots + a_rw_r = a_1T(u_1) + a_2T(u_2) + \ldots + a_rT(u_r) = T(a_1u_1 + a_2u_2 + \ldots + a_ru_r)$$

Therefore, $T(v - a_1u_1 - a_2u_2 - \ldots - a_ru_r) = 0$, i.e.,

$$v - a_1u_1 - a_2u_2 - \ldots - a_ru_r \in K(T).$$

It follows that

$$v - a_1u_1 - a_2u_2 - \ldots - a_ru_r = b_1v_1 + b_2v_2 + \ldots + b_kv_k$$

$$\Rightarrow v = a_1u_1 + a_2u_2 + \ldots + a_ru_r + b_1v_1 + b_2v_2 + \ldots + b_kv_k$$
Proof of $V = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$

For every $\mathbf{v} \in V$, $T(\mathbf{v}) \in R(T)$ and hence

$$T(\mathbf{v}) = a_1 \mathbf{w}_1 + a_2 \mathbf{w}_2 + \ldots + a_r \mathbf{w}_r$$

$$= a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + \ldots + a_r T(\mathbf{u}_r)$$

$$= T(a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \ldots + a_r \mathbf{u}_r)$$

Therefore, $T(\mathbf{v} - a_1 \mathbf{u}_1 - a_2 \mathbf{u}_2 - \ldots - a_r \mathbf{u}_r) = 0$, i.e.,

$$\mathbf{v} - a_1 \mathbf{u}_1 - a_2 \mathbf{u}_2 - \ldots - a_r \mathbf{u}_r \in K(T).$$

It follows that

$$\mathbf{v} - a_1 \mathbf{u}_1 - a_2 \mathbf{u}_2 - \ldots - a_r \mathbf{u}_r = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_k \mathbf{v}_k$$

$$\Rightarrow \mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \ldots + a_r \mathbf{u}_r + b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_k \mathbf{v}_k$$
Proof of $V = \text{Span}\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_k\}$

For every $v \in V$, $T(v) \in R(T)$ and hence

$$T(v) = a_1w_1 + a_2w_2 + \ldots + a_rw_r$$
$$= a_1T(u_1) + a_2T(u_2) + \ldots + a_rT(u_r)$$
$$= T(a_1u_1 + a_2u_2 + \ldots + a_ru_r)$$

Therefore, $T(v - a_1u_1 - a_2u_2 - \ldots - a_ru_r) = 0$, i.e.,

$$v - a_1u_1 - a_2u_2 - \ldots - a_ru_r \in K(T).$$

It follows that

$$v - a_1u_1 - a_2u_2 - \ldots - a_ru_r = b_1v_1 + b_2v_2 + \ldots + b_kv_k$$
$$\Rightarrow v = a_1u_1 + a_2u_2 + \ldots + a_ru_r + b_1v_1 + b_2v_2 + \ldots + b_kv_k$$
Otherwise,

$$a_1 u_1 + a_2 u_2 + \ldots + a_r u_r + b_1 v_1 + b_2 v_2 + \ldots + b_k v_k = 0$$

for some $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_k \in \mathbb{R}$, not all zero. Then

$$a_1 \underbrace{T(u_1)}_{w_1} + a_2 \underbrace{T(u_2)}_{w_2} + \ldots + a_r \underbrace{T(u_r)}_{w_r} + \underbrace{T(b_1 v_1 + b_2 v_2 + \ldots + b_k v_k)}_{0} = 0$$

And since $w_1, w_2, \ldots, w_r$ are linearly independent,

$$a_1 = a_2 = \ldots = a_r = 0.$$ Therefore,

$$b_1 v_1 + b_2 v_2 + \ldots + b_k v_k = 0.$$ And since $v_1, v_2, \ldots, v_k$ are linearly independent,

$$b_1 = b_2 = \ldots = b_k = 0.$$ Contradiction.
Otherwise,

\[ a_1 u_1 + a_2 u_2 + ... + a_r u_r + b_1 v_1 + b_2 v_2 + ... + b_k v_k = 0 \]

for some \( a_1, a_2, ..., a_r, b_1, b_2, ..., b_k \in \mathbb{R} \), not all zero. Then

\[
\begin{align*}
    a_1 \begin{bmatrix} \mathbf{w}_1 \\ T(\mathbf{u}_1) \end{bmatrix} + a_2 \begin{bmatrix} \mathbf{w}_2 \\ T(\mathbf{u}_2) \end{bmatrix} + ... + a_r \begin{bmatrix} \mathbf{w}_r \\ T(\mathbf{u}_r) \end{bmatrix} + T\left( b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + ... + b_k \mathbf{v}_k \right) &= 0
\end{align*}
\]

And since \( \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_r \) are linearly independent, \( a_1 = a_2 = ... = a_r = 0 \). Therefore,

\[ b_1 v_1 + b_2 v_2 + ... + b_k v_k = 0. \]

And since \( \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \) are linearly independent, \( b_1 = b_2 = ... = b_k = 0 \). Contradiction.
Linear Independence of \( u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_k \)

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for some \( a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_k \in \mathbb{R} \), not all zero. Then

\[
    a_1 T(u_1) + a_2 T(u_2) + \ldots + a_r T(u_r) + T(b_1 v_1 + b_2 v_2 + \ldots + b_k v_k) = 0
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And since \( w_1, w_2, \ldots, w_r \) are linearly independent,

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for some \( a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_k \in \mathbb{R} \), not all zero. Then

\[ \underbrace{T(u_1)}_{w_1} + \underbrace{T(u_2)}_{w_2} + \ldots + \underbrace{T(u_r)}_{w_r} + \underbrace{T(b_1v_1 + b_2v_2 + \ldots + b_kv_k)}_{0} = 0 \]

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Linear Independence of $u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_k$

Otherwise,

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for some $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_k \in \mathbb{R}$, not all zero. Then

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And since $w_1, w_2, \ldots, w_r$ are linearly independent, $a_1 = a_2 = \ldots = a_r = 0$. Therefore,

$$b_1 v_1 + b_2 v_2 + \ldots + b_k v_k = 0.$$ 

And since $v_1, v_2, \ldots, v_k$ are linearly independent, $b_1 = b_2 = \ldots = b_k = 0$. Contradiction.
Let $T : V \rightarrow W$ be a linear transformation between two vector spaces of dimensions $\dim V = n$ and $\dim W = m$ and let $A = [T]$ be the matrix representing $T$. Then

$$T \text{ 1-1 } \iff K(T) = \{0\} \iff \text{Nul}(A) = \{0\}.$$ 

$$T \text{ onto } \iff R(T) = W \iff \text{rank}(T) = \dim W \iff \text{rank}(A) = m.$$ 

$T$ bijective $\iff T$ 1-1 and onto $\iff K(T) = \{0\}, \text{rank}(T) = m$ 

$A$ invertible $\iff \text{Nul}(A) = \{0\}, \text{rank}(A) = m$
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$A$ invertible
Corollary

Let $A$ be an $m \times n$ matrix. Then

- $\text{rank}(A) + \dim \text{Nul}(A) = n$.
- The solution set of $Ax = 0$ is a subspace of $\mathbb{R}^n$ of dimension $n - \text{rank}(A)$. In particular, if $m < n$, $Ax = 0$ has infinitely many solutions.

- $Ax = b$ has a solution if and only if $b \in \text{Col}(A)$, or equivalently, $\text{rank}(A) = \text{rank} \begin{bmatrix} A & b \end{bmatrix}$. If $x_0$ is a solution of $Ax = b$, then

$$\{ x : Ax = b \} = x_0 + \text{Nul}(A) = \{ x_0 + v : v \in \text{Nul}(A) \}.$$  

In particular, if $m < n$, $Ax = b$ has either no solutions or infinitely many solutions.
Corollary

Let $A$ be an $m \times n$ matrix. Then

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In particular, if $m < n$, $Ax = b$ has either no solutions or infinitely many solutions.
Consequences of Rank Theorem

Corollary

Let $A$ be an $m \times n$ matrix. Then

- $\text{rank}(A) + \text{dim} \text{Null}(A) = n$.
- The solution set of $Ax = 0$ is a subspace of $\mathbb{R}^n$ of dimension $n - \text{rank}(A)$. In particular, if $m < n$, $Ax = 0$ has infinitely many solutions.
- $Ax = b$ has a solution if and only if $b \in \text{Col}(A)$, or equivalently, $\text{rank}(A) = \text{rank} \begin{bmatrix} A & b \end{bmatrix}$. If $x_0$ is a solution of $Ax = b$, then

$$\{ x : Ax = b \} = x_0 + \text{Null}(A) = \{ x_0 + v : v \in \text{Null}(A) \}.$$

In particular, if $m < n$, $Ax = b$ has either no solutions or infinitely many solutions.
Proof.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $T(x) = Ax$. Then

$$[T] = A, \dim K(T) = \dim \text{Nul}(A),$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$

$Ax = b$ has a solution if and only if $T(x) = b$ for some $x$, i.e.,

$$b \in R(T) \iff b \in \text{Col}(A)$$

$$\iff \text{Col}(A) = \text{Col}[A \ b] \iff \text{rank}(A) = \text{rank}[A \ b].$$

Finally, if $x_0$ and $x_1$ are two solutions of $Ax = b$. Then

$$T(x_1 - x_0) = T(x_1) - T(x_0) = b - b = 0,$$ i.e.,

$$x_1 - x_0 \in K(T) = \text{Nul}(A) \Rightarrow x_1 \in x_0 + \text{Nul}(A).$$
Consequences of Rank Theorem

Proof.
Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the linear transformation given by $T(x) = Ax$. Then

$$[T] = A, \dim K(T) = \dim \text{Nul}(A),$$

$$\text{rank}(T) = \dim R(T) = \dim \text{Col}(A) = \text{rank}(A).$$

$Ax = b$ has a solution if and only if $T(x) = b$ for some $x$, i.e.,

$$b \in R(T) \iff b \in \text{Col}(A)$$

$$\iff \text{Col}(A) = \text{Col} \begin{bmatrix} A & b \end{bmatrix} \iff \text{rank}(A) = \text{rank} \begin{bmatrix} A & b \end{bmatrix}.$$  

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Rank of a Linear Transformation
Consequences of Rank Theorem

**Consequences of Rank Theorem**

**Proof.** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be the linear transformation given by \( T(x) = Ax \). Then
\[
[T] = A, \quad \dim K(T) = \dim \operatorname{Nul}(A), \quad \text{rank}(T) = \dim R(T) = \dim \operatorname{Col}(A) = \text{rank}(A).
\]

\( Ax = b \) has a solution if and only if \( T(x) = b \) for some \( x \), i.e.,
\[
b \in R(T) \iff b \in \operatorname{Col}(A) \iff \operatorname{Col}(A) = \operatorname{Col}[A \ b] \iff \text{rank}(A) = \text{rank} [A \ b].
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T(x_1 - x_0) = T(x_1) - T(x_0) = b - b = 0, \quad \text{i.e.,} \quad x_1 - x_0 \in K(T) = \operatorname{Nul}(A) \Rightarrow x_1 \in x_0 + \operatorname{Nul}(A).
\]
Corollary

Let $A$ be an $n \times n$ matrix. Then the following are equivalent:

- $A$ is nonsingular (det $A \neq 0$).
- $A$ is invertible ($A$ has an inverse $A^{-1}$ satisfying $AA^{-1} = A^{-1}A = I$).
- Nul($A$) = \{0\}.
- $Ax = 0$ has only one solution.
- rank($A$) = $n$.
- $Ax = b$ has a solution for all $b \in \mathbb{R}^n$. 
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Xi Chen

Linear Algebra II Lecture 15
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Proof.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $T(x) = Ax$. Then $[T] = A$ and

$$\det A \neq 0 \iff A \text{ invertible} \iff T \text{ bijective}$$

$$\text{Nul}(A) = \{0\} \iff T \text{ injective}$$

$$\text{rank}(A) = n \iff T \text{ onto}$$

where CR = Cramer’s Rule and RT = Rank Theorem.