Linear Algebra II Lecture 13

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Outline

1. Injection, Surjection and Isomorphism

2. Lagrange Interpolation
Definition

**Definition of Injection, Surjection and Bijection**

We call a map \( f : X \rightarrow Y \) an injection (injective, one-to-one, 1-1) if \( f(x_1) \neq f(x_2) \) for all \( x_1 \neq x_2 \in X \).

We call a map \( f : X \rightarrow Y \) a surjection (surjective, onto) if \( f(X) = Y \), i.e., for every \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \).

We call a map \( f : X \rightarrow Y \) a bijection (bijective) if it is 1-1 and onto. A bijection \( f : X \rightarrow Y \) has an inverse \( f^{-1} : Y \rightarrow X \) such that \( f \circ f^{-1} = 1_Y \) and \( f^{-1} \circ f = 1_X \), where \( 1_X \) and \( 1_Y \) are the identity maps on \( X \) and \( Y \).

**Isomorphism**

A linear transformation \( T : V \rightarrow W \) is an isomorphism if it is an bijection. Two vector spaces are isomorphic (written as \( V \cong W \)) if there is an isomorphism \( T : V \rightarrow W \).
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Theorem
Let $T : V \rightarrow W$ be a linear transformation. Then

- $T$ is 1-1 if and only if $K(T) = \{0\}$;
- $T$ is onto if and only if $R(T) = W$.

Proof.
If $T$ is 1-1, then $T(v) \neq T(0) = 0$ for all $v \neq 0$. Therefore, $v \notin K(T)$ for all $v \neq 0$. That is, $K(T) = \{0\}$.

Suppose that $K(T) = \{0\}$. If $T$ is not 1-1, then there exists $v_1 \neq v_2$ such that $T(v_1) = T(v_2)$. Then

$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0$$

and $v_1 - v_2 \neq 0 \in K(T)$. Contradiction.
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T(\mathbf{v}_1 - \mathbf{v}_2) = T(\mathbf{v}_1) - T(\mathbf{v}_2) = 0
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and \( \mathbf{v}_1 - \mathbf{v}_2 \neq 0 \in K(T) \). Contradiction.
Let $T : V \rightarrow W$ be an isomorphism. Then $T^{-1}$ is also a linear transformation.

In addition, if $V$ is finite dimensional, $\dim V = \dim W$ and

$$[T]^{-1}_{B,C} = [T^{-1}]_{C,B}.$$
Inverse Linear Transformation

**Theorem**

Let $T : V \rightarrow W$ be an isomorphism. Then $T^{-1}$ is also a linear transformation. In addition, if $V$ is finite dimensional, $\dim V = \dim W$ and

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Theorem

Let \( T : V \rightarrow W \) be an isomorphism. Then \( T^{-1} \) is also a linear transformation.
In addition, if \( V \) is finite dimensional, \( \dim V = \dim W \) and

\[
[T]_{B,C}^{-1} = [T^{-1}]_{C,B}.
\]

\( T^{-1} \) is a linear transformation.

For \( w_1, w_2 \in W \), let \( v_1 = T^{-1}(w_1) \) and \( v_2 = T^{-1}(w_2) \). Then

\[
T(v_1 + cv_2) = T(v_1) + cT(v_2) = w_1 + cw_2
\]

\[\Rightarrow\]

\[
\underbrace{T^{-1}(w_1)} + c \underbrace{T^{-1}(w_2)} = T^{-1}(w_1 + cw_2)
\]
Proof.

If \( \dim V = n < \infty \), let \( B = \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \). Then

\[
R(T) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.
\]

Since \( T \) is onto, \( R(T) = W \) and hence \( \dim W = m \leq n \). Let \( C = \{w_1, w_2, \ldots, w_m\} \) be a basis of \( W \). Then

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Since \( T^{-1} \) is onto, \( R(T^{-1}) = V \) and hence \( \dim V = n \leq m \). So \( m = n \). Finally,

\[
T \circ T^{-1} = 1_W \Rightarrow [T]_{B,C} [T^{-1}]_{C,B} = [1_W]_{C,C} = I
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Question

Given $n$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, assuming that $x_1, x_2, \ldots, x_n$ are distinct, find a polynomial $f(x)$ of degree $\leq n - 1$ such that the curve $y = f(x)$ passes through these $n$ points, i.e. $f(x_k) = y_k$ for $k = 1, 2, \ldots, n$.

Let $f(x) = a_1 + a_2x + \ldots + a_nx^{n-1}$. Then it comes down to solve

$$
\begin{align*}
&f(x_1) = y_1 \\
f(x_2) = y_2 \\
&\vdots \\
f(x_n) = y_n
\end{align*}
\iff
\begin{bmatrix}
1 & x_1 & \ldots & x_1^{n-1} \\
1 & x_2 & \ldots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \ldots & x_n^{n-1}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
$$

Xi Chen
Linear Algebra II Lecture 13
Interpolation Problem

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Let $f(x) = a_1 + a_2 x + \ldots + a_n x^{n-1}$. Then it comes down to solve

$$
\begin{align*}
\begin{cases}
f(x_1) = y_1 \\
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\vdots \\
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\end{cases} &\iff 
\begin{bmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\
1 & x_2 & \ldots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \ldots & x_n^{n-1} \end{bmatrix}
\begin{bmatrix} a_1 \\
a_2 \\
\vdots \\
a_n \end{bmatrix} =
\begin{bmatrix} y_1 \\
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Interpolation Problem

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Given \( n \) points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), assuming that \( x_1, x_2, \ldots, x_n \) are distinct, find a polynomial \( f(x) \) of degree \( \leq n - 1 \) such that the curve \( y = f(x) \) passes through these \( n \) points, i.e. \( f(x_k) = y_k \) for \( k = 1, 2, \ldots, n \).

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  \vdots & \quad \vdots \\
  f(x_n) &= y_n \\
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\]

\[
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  \vdots & \vdots & \ddots & \vdots \\
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  a_1 \\
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\end{bmatrix}
= 
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
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\end{bmatrix}
\]
An Approach Using Linear Transformation

Let $V = \{ f(x) \in \mathbb{R}[x] : \deg f(x) \leq n - 1 \}$ and let $T : V \to \mathbb{R}^n$ be the linear transformation

$$T(f(x)) = (f(x_1), f(x_2), \ldots, f(x_n)).$$

Objective: find $f(x) \in V$ such that

$$T(f(x)) = (y_1, y_2, \ldots, y_n)$$

or equivalently, find

$$T^{-1}(y_1, y_2, \ldots, y_n).$$

It suffices to find $T^{-1}(e_k)$ since

$$T^{-1}(y_1, y_2, \ldots, y_n) = y_1 T^{-1}(e_1) + y_2 T^{-1}(e_2) + \ldots + y_n T^{-1}(e_n).$$
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Find $T^{-1}(e_k)$

Let $f_k(x) = T^{-1}(e_k)$. Then $T(f_k(x)) = e_k$, i.e.,

$$T(f_k(x)) = (f_k(x_1), f_k(x_2), ..., f_k(x_n)) = e_k$$

$$\Rightarrow f_k(x_k) = 1 \text{ and } f_k(x_i) = 0 \text{ for } i \neq k$$

$$\Rightarrow f_k(x_1) = f_k(x_2) = ... = f_k(x_{k-1}) = f_k(x_{k+1}) = ... = f_k(x_n) = 0$$

$$f(x) = c(x - x_1)(x - x_2)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)$$

$$\Rightarrow \quad = c \prod_{i \neq k}(x - x_i)$$

for some $c \in \mathbb{R}$. 
Find $T^{-1}(e_k)$

Let $f_k(x) = T^{-1}(e_k)$. Then $T(f_k(x)) = e_k$, i.e.,

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$$f(x) = c(x - x_1)(x - x_2)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)$$

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for some $c \in \mathbb{R}$. 
Find $T^{-1}(e_k)$

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$$f(x) = c(x - x_1)(x - x_2)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)$$

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Find $T^{-1}(\mathbf{e}_k)$

Let $f_k(x) = T^{-1}(\mathbf{e}_k)$. Then $T(f_k(x)) = \mathbf{e}_k$, i.e.,

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for some $c \in \mathbb{R}$.
Find $c$

\[ f_k(x_k) = 1 \Rightarrow c \prod_{i \neq k} (x_k - x_i) = 1 \Rightarrow c = \prod_{i \neq k} (x_k - x_i)^{-1} \]

\[ f_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i} \]

\[ = \frac{(x - x_1)(x - x_2)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_1)(x_k - x_2)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)} \]

for $k = 1, 2, \ldots, n$. Finally,

\[ T^{-1}(y_1, y_2, \ldots, y_n) = y_1 f_1(x) + y_2 f_2(x) + \ldots + y_n f_n(x) \]

where the RHS is called the Lagrange interpolating polynomial passing through $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. 
Find $c$

\[
f_k(x_k) = 1 \Rightarrow c \prod_{i \neq k}(x_k - x_i) = 1 \Rightarrow c = \prod_{i \neq k}(x_k - x_i)^{-1}
\]

\[
f_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i}
\]

\[
\Rightarrow \frac{(x - x_1)(x - x_2)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_1)(x_k - x_2)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}
\]

for $k = 1, 2, \ldots, n$. Finally,

\[
T^{-1}(y_1, y_2, \ldots, y_n) = y_1 f_1(x) + y_2 f_2(x) + \ldots + y_n f_n(x)
\]

where the RHS is called the Lagrange interpolating polynomial passing through $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. 
Find $c$

$$f_k(x_k) = 1 \Rightarrow c \prod_{i \neq k} (x_k - x_i) = 1 \Rightarrow c = \prod_{i \neq k} (x_k - x_i)^{-1}$$

$$f_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i}$$

$$\Rightarrow \quad \frac{(x - x_1)(x - x_2) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_n)}{(x_k - x_1)(x_k - x_2) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_n)}$$

for $k = 1, 2, \ldots, n$. Finally,

$$T^{-1}(y_1, y_2, \ldots, y_n) = y_1 f_1(x) + y_2 f_2(x) + \ldots + y_n f_n(x)$$

where the RHS is called the *Lagrange interpolating polynomial* passing through $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. 