Kernel as Null Space and Range as Column Space
Injection, Surjection and Isomorphism

Linear Algebra II Lecture 12

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Outline

1. Kernel as Null Space and Range as Column Space

2. Injection, Surjection and Isomorphism
Recall that $K(T) = \{ \mathbf{v} : T(\mathbf{v}) = 0 \}$. If $T$ is represented by a matrix $A = [T]$, then $T(\mathbf{v}) = A\mathbf{v}$, roughly. So

$$K(T) = \text{Nul}(A).$$

Recall that $R(T) = T(V) = \{ T(\mathbf{v}) : \mathbf{v} \in V \}$ and hence

$$R(T) = \text{Span}\{ T(\mathbf{v}) : \mathbf{v} \in B \}$$

for a basis $B$. If $T$ is represented by a matrix $A = [T]$, then

$$R(T) = \text{Col}(A).$$
Notations $[\cdot]_B$ and $[\cdot]^B$

Given a vector space $V$ and an ordered basis $B = \{v_1, v_2, ..., v_n\}$ of $V$, every $v$ can expressed as a linear combination of $B$ in a unique way

$$v = x_1 v_1 + x_2 v_2 + ... + x_n v_n$$

Then we write

$$[v]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^B$

In other words, $[\cdot]_B : V \rightarrow \mathbb{R}^n$ is the map sending $v_i$ to $e_i$ and $[\cdot]^B : \mathbb{R}^n \rightarrow V$ is its inverse.
Notations $[\cdot]_B$ and $[\cdot]^B$

For example, let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis in $\mathbb{R}^n$. Then

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}_B = A^{-1} \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
$$

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}^B = A \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
$$

where $A = [v_1 \ v_2 \ \ldots \ v_n]$ is the matrix with column vectors $v_k$. 
Let $B = \{(1, 2), (2, 3)\}$ in $\mathbb{R}^2$. Then

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

i.e., $(1, 1) = -(1, 2) + (2, 3)$. On the other hand,

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}^B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
**Theorem**

Let $T : V \to W$ be a linear transformation between two vector spaces of finite dimensions and let $[T] = [T]_{B,C}$ be the matrix representing $T$ under the bases $B$ and $C$ of $V$ and $W$, respectively. Then $v \in K(T)$ if and only if $[v]_B \in \text{Nul}([T])$. That is, $K(T) = [\text{Nul}([T])]^B$.

**Proof.**

Since

$$T(v) = \left([T]_{B,C}[v]_B\right)^C$$

$$T(v) = 0 \iff [T][v]_B = 0 \iff [v]_B \in \text{Nul}([T]).$$
Theorem

Let \( T : V \rightarrow W \) be a linear transformation between two vector spaces of finite dimensions and let \([ T ] = [ T ]_{B,C}\) be the matrix representing \( T \) under the bases \( B \) and \( C \) of \( V \) and \( W \), respectively. Then \( w \in R(T) \) if and only if \([ w ]_C \in \text{Col}([ T ])\). That is, \( R(T) = [ \text{Col}([ T ]) ]^C \).

Proof.

Let \( B = \{ v_1, v_2, \ldots, v_n \} \). Then

\[
\begin{align*}
  w \in R(T) & \iff w \in \text{Span}\{ T(v_1), T(v_2), \ldots, T(v_n) \} \\
  & \iff [ w ]_C \in \text{Span}\{ [ T(v_1) ]_C, [ T(v_2) ]_C, \ldots, [ T(v_n) ]_C \}
\end{align*}
\]
Example of $K(T) = \text{Nul}(\begin{bmatrix} T \end{bmatrix})$ and $R(T) = \text{Col}(\begin{bmatrix} T \end{bmatrix})$

Let $V = \{ f(x) \in \mathbb{R}[x] : \deg f(x) \leq n \}$ and $T$ be the map given by

$$T(f(x)) = f'(x).$$

Let $B = \{1, x, \ldots, x^n\}$. Then

$$ A = [T] = [T]_{B,B} = \begin{bmatrix} 0 & 1 & & & \\ 0 & 2 & & & \\ & \ddots & \ddots & \vdots \\ & & 0 & n \\ & & & 0 \\ & & & & 0 \end{bmatrix}$$
Example of $K(T) = \text{Nul}([T])$ and $R(T) = \text{Col}([T])$

We know that

$$K(T) = \{f(x) : f'(x) \equiv 0 \} = \{c \in \mathbb{R}\} = \text{Span}\{1\}.$$  

On the other hand,

$$K(T) = [\text{Nul}(A)]^B = [\text{Span}\{e_1\}]^B = \text{Span}\{[e_1]^B\} = \text{Span}\{1\}.$$  

For every polynomial $f(x)$ of degree $\leq n - 1$, there exists $F(x) \in V$ such that

$$F'(x) = f(x).$$

Therefore,

$$R(T) = \{f(x) \in \mathbb{R}[x] : \text{deg } f(x) \leq n - 1\} = \text{Span}\{1, x, \ldots, x^{n-1}\}.$$
Example of $K(T) = \text{Nul}([T])$ and $R(T) = \text{Col}([T])$

On the other hand,

$$R(T) = \left[ \text{Col}(A) \right]^B = \left[ \text{Span}\{0, e_1, 2e_2, \ldots, ne_n\} \right]^B$$

$$= \text{Span}\{[e_1]^B, [e_2]^B, \ldots, [e_n]^B\}$$

$$= \text{Span}\{1, x, \ldots, x^{n-1}\}$$
**Definition of Injection, Surjection and Bijection**

We call a map \( f : X \to Y \) an injection (injective, one-to-one, 1-1) if \( f(x_1) \neq f(x_2) \) for all \( x_1 \neq x_2 \in X \).

We call a map \( f : X \to Y \) a surjection (surjective, onto) if \( f(X) = Y \), i.e., for every \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \).

We call a map \( f : X \to Y \) a bijection (bijective) if it is 1-1 and onto. A bijection \( f : X \to Y \) has an inverse \( f^{-1} : Y \to X \) such that \( f \circ f^{-1} = 1_Y \) and \( f^{-1} \circ f = 1_X \), where \( 1_X \) and \( 1_Y \) are the identity maps on \( X \) and \( Y \).

**Isomorphism**

A linear transformation \( T : V \to W \) is an isomorphism if it is an bijection. Two vector spaces are isomorphic (written as \( V \cong W \)) if there is an isomorphism \( T : V \to W \).
Injective and Surjective Linear Transformations

**Theorem**

Let $T : V \rightarrow W$ be a linear transformation. Then

- $T$ is 1-1 if and only if $K(T) = \{0\}$;
- $T$ is onto if and only if $R(T) = W$.

**Proof.**

If $T$ is 1-1, then $T(v) \neq T(0) = 0$ for all $v \neq 0$. Therefore, $v \notin K(T)$ for all $v \neq 0$. That is, $K(T) = \{0\}$.

Suppose that $K(T) = \{0\}$. If $T$ is not 1-1, then there exists $v_1 \neq v_2$ such that $T(v_1) = T(v_2)$. Then

$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0$$

and $v_1 - v_2 \neq 0 \in K(T)$. Contradiction.
Theorem

Let $T : V \rightarrow W$ be an isomorphism. Then $T^{-1}$ is also a linear transformation.
In addition, if $V$ is finite dimensional, $\dim V = \dim W$ and

$$[T]_{B,C}^{-1} = [T^{-1}]_{C,B}.$$

$T^{-1}$ is a linear transformation.

For $w_1, w_2 \in W$, let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Then

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = w_1 + cw_2$$

$$\Rightarrow \underbrace{v_1 + c}_T^{-1}(w_1) \underbrace{v_2 +}_T^{-1}(w_2) = T^{-1}(w_1 + cw_2)$$
Proof.

If \( \dim V = n < \infty \), let \( B = \{v_1, v_2, ..., v_n\} \) be a basis of \( V \). Then

\[
R(T) = \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}.
\]

Since \( T \) is onto, \( R(T) = W \) and hence \( \dim W = m \leq n \). Let \( C = \{w_1, w_2, ..., w_m\} \) be a basis of \( W \). Then

\[
R(T^{-1}) = \text{Span}\{T^{-1}(w_1), T^{-1}(w_2), ..., T^{-1}(w_m)\}.
\]

Since \( T^{-1} \) is onto, \( R(T^{-1}) = V \) and hence \( \dim V = n \leq m \). So \( m = n \). Finally,

\[
T \circ T^{-1} = 1_W \Rightarrow [T]_{B,C}[T^{-1}]_{C,B} = [1_W]_{C,C} = I
\Rightarrow [T]^{-1}_{B,C} = [T^{-1}]_{C,B}
\]