Linear Algebra II Lecture 12

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Outline

1. Kernel as Null Space and Range as Column Space

2. Injection, Surjection and Isomorphism
Recall that $K(T) = \{ \mathbf{v} : T(\mathbf{v}) = 0 \}$. If $T$ is represented by a matrix $A = [T]$, then $T(\mathbf{v}) = A\mathbf{v}$, roughly. So

$$K(T) = \text{Nul}(A).$$

Recall that $R(T) = T(V) = \{ T(\mathbf{v}) : \mathbf{v} \in V \}$ and hence

$$R(T) = \text{Span}\{ T(\mathbf{v}) : \mathbf{v} \in B \}$$

for a basis $B$. If $T$ is represented by a matrix $A = [T]$, then

$$R(T) = \text{Col}(A).$$
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$$R(T) = \text{Col}(A).$$
$K(T) = \text{Nul}([T])$ and $R(T) = \text{Col}([T])$

Recall that $K(T) = \{ v : T(v) = 0 \}$. If $T$ is represented by a matrix $A = [T]$, then $T(v) = Av$, roughly. So

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for a basis $B$. If $T$ is represented by a matrix $A = [T]$, then

$$R(T) = \text{Col}(A).$$
Given a vector space $V$ and an ordered basis $B = \{v_1, v_2, \ldots, v_n\}$ of $V$, every $v$ can be expressed as a linear combination of $B$ in a unique way

$$v = x_1v_1 + x_2v_2 + \ldots + x_nv_n$$

Then we write

$$[v]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^B$$

In other words, $[]_B : V \to \mathbb{R}^n$ is the map sending $v_i$ to $e_i$ and $[]^B : \mathbb{R}^n \to V$ is its inverse.
Given a vector space $V$ and an ordered basis $B = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}$ of $V$, every $\mathbf{v}$ can expressed as a linear combination of $B$ in a unique way

$$\mathbf{v} = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \ldots + x_n \mathbf{v}_n$$

Then we write

$$[\mathbf{v}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^B$$

In other words, $[\cdot]_B : V \rightarrow \mathbb{R}^n$ is the map sending $\mathbf{v}_i$ to $\mathbf{e}_i$ and $[\cdot]^B : \mathbb{R}^n \rightarrow V$ is its inverse.
Given a vector space $V$ and an ordered basis $B = \{v_1, v_2, \ldots, v_n\}$ of $V$, every $v$ can expressed as a linear combination of $B$ in a unique way

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In other words, $[\cdot]_B : V \to \mathbb{R}^n$ is the map sending $v_i$ to $e_i$ and $[]^B : \mathbb{R}^n \to V$ is its inverse.
For example, let \( B = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) be a basis in \( \mathbb{R}^n \). Then

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}_B = A^{-1} \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}_B = A \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
\]

where \( A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} \) is the matrix with column vectors \( \mathbf{v}_k \).
For example, let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis in $\mathbb{R}^n$. Then

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}_B = A^{-1}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}_B = A
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
$$

where $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$ is the matrix with column vectors $v_k$. 
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x_2 \\
\vdots \\
x_n
\end{bmatrix}
$$

$$
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x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}_B = A
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
$$

where $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$ is the matrix with column vectors $v_k$. 
Let $B = \{(1, 2), (2, 3)\}$ in $\mathbb{R}^2$. Then

$$
\begin{bmatrix}
1 \\
1
\end{bmatrix}_B = \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix}^{-1} \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
-3 & 2 \\
2 & -1
\end{bmatrix} \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
-1 \\
1
\end{bmatrix}
$$

i.e., $(1, 1) = -(1, 2) + (2, 3)$. On the other hand,

$$
\begin{bmatrix}
-1 \\
1
\end{bmatrix}^B = \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix} \begin{bmatrix}
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix}
$$
Let \( B = \{(1, 2), (2, 3)\} \) in \( \mathbb{R}^2 \). Then

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1
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1
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\begin{bmatrix}
-1 \\
1
\end{bmatrix} = 
\begin{bmatrix}
1
\end{bmatrix}
\]
Theorem

Let \( T : V \rightarrow W \) be a linear transformation between two vector spaces of finite dimensions and let \([T] = [T]_{B,C}\) be the matrix representing \( T \) under the bases \( B \) and \( C \) of \( V \) and \( W \), respectively. Then \( \mathbf{v} \in K(T) \) if and only if \([\mathbf{v}]_B \in \text{Nul}([T])\). That is, \( K(T) = \left[ \text{Nul}([T]) \right]^B \).

Proof.

Since

\[
T(\mathbf{v}) = \left[ [T]_{B,C} [\mathbf{v}]_B \right]^C
\]

\[
T(\mathbf{v}) = 0 \iff [T][\mathbf{v}]_B = 0 \iff [\mathbf{v}]_B \in \text{Nul}([T]).
\]
**Theorem**

Let $T : V \rightarrow W$ be a linear transformation between two vector spaces of finite dimensions and let $[T] = [T]_{B,C}$ be the matrix representing $T$ under the bases $B$ and $C$ of $V$ and $W$, respectively. Then $v \in K(T)$ if and only if $[v]_B \in \text{Nul}([T])$. That is, $K(T) = [\text{Nul}([T])]_B$.

**Proof.**

Since

$$T(v) = [T]_{B,C}[v]_B^C$$

$$T(v) = 0 \iff [T][v]_B = 0 \iff [v]_B \in \text{Nul}([T]).$$
Theorem

Let $T : V \to W$ be a linear transformation between two vector spaces of finite dimensions and let $[T] = [T]_{B,C}$ be the matrix representing $T$ under the bases $B$ and $C$ of $V$ and $W$, respectively. Then $v \in K(T)$ if and only if $[v]_B \in \text{Nul}([T])$. That is, $K(T) = \left[ \text{Nul}([T]) \right]^B$.

Proof.

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Theorem

Let $T : V \rightarrow W$ be a linear transformation between two vector spaces of finite dimensions and let $[T] = [T]_{B,C}$ be the matrix representing $T$ under the bases $B$ and $C$ of $V$ and $W$, respectively. Then $w \in R(T)$ if and only if $[w]_C \in \text{Col}([T])$. That is, $R(T) = [\text{Col}([T])]^C$.

Proof.

Let $B = \{v_1, v_2, ..., v_n\}$. Then

\[ w \in R(T) \iff w \in \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\} \]
\[ \iff [w]_C \in \text{Span}\{[T(v_1)]_C, [T(v_2)]_C, ..., [T(v_n)]_C\} \]
Theorem

Let $T : V \to W$ be a linear transformation between two vector spaces of finite dimensions and let $[T] = [T]_{B,C}$ be the matrix representing $T$ under the bases $B$ and $C$ of $V$ and $W$, respectively. Then $w \in R(T)$ if and only if $[w]_C \in \text{Col}([T])$. That is, $R(T) = \left[ \text{Col}([T]) \right]^C$.

Proof.

Let $B = \{v_1, v_2, ..., v_n\}$. Then

$$w \in R(T) \iff w \in \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}$$

$$\iff [w]_C \in \text{Span}\{[T(v_1)]_C, [T(v_2)]_C, ..., [T(v_n)]_C\}$$
Example of $K(T) = \text{Nul}([T])$ and $R(T) = \text{Col}([T])$

Let $V = \{f(x) \in \mathbb{R}[x] : \text{deg } f(x) \leq n\}$ and $T$ be the map given by

$$T(f(x)) = f'(x).$$

Let $B = \{1, x, ..., x^n\}$. Then

$$A = [T] = [T]_{B,B} = \begin{bmatrix} 0 & 1 & 0 & 2 & \cdots & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \end{bmatrix}$$
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$$A = [T] = [T]_{B,B} = \begin{bmatrix} 0 & 1 & & \\ 0 & 2 & & \\ & & \ddots & \ddots \\ & & & 0 & n \end{bmatrix}$$
We know that

\[ K(T) = \{ f(x) : f'(x) \equiv 0 \} = \{ c \in \mathbb{R} \} = \text{Span}\{1\}. \]

On the other hand,

\[ K(T) = [\text{Nul}(A)]^B = [\text{Span}\{e_1\}]^B = \text{Span}\{[e_1]^B\} = \text{Span}\{1\}. \]

For every polynomial \( f(x) \) of degree \( \leq n - 1 \), there exists \( F(x) \in V \) such that

\[ F'(x) = f(x). \]

Therefore,

\[ R(T) = \{ f(x) \in \mathbb{R}[x] : \deg f(x) \leq n - 1 \} = \text{Span}\{1, x, \ldots, x^{n-1}\}. \]
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Example of $K(T) = \text{Nul}([T])$ and $R(T) = \text{Col}([T])$

On the other hand,

$$R(T) = [\text{Col}(A)]^B = [\text{Span}\{0, e_1, 2e_2, \ldots, ne_n\}]^B$$

$$= \text{Span}\{[e_1]^B, [e_2]^B, \ldots, [e_n]^B\}$$

$$= \text{Span}\{1, x, \ldots, x^{n-1}\}$$
Definition of Injection, Surjection and Bijection

We call a map \( f : X \to Y \) an injection (injective, one-to-one, 1-1) if \( f(x_1) \neq f(x_2) \) for all \( x_1 \neq x_2 \in X \).

We call a map \( f : X \to Y \) a surjection (surjective, onto) if \( f(X) = Y \), i.e., for every \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \).

We call a map \( f : X \to Y \) a bijection (bijective) if it is 1-1 and onto. A bijection \( f : X \to Y \) has an inverse \( f^{-1} : Y \to X \) such that \( f \circ f^{-1} = 1_Y \) and \( f^{-1} \circ f = 1_X \), where \( 1_X \) and \( 1_Y \) are the identity maps on \( X \) and \( Y \).

Isomorphism

A linear transformation \( T : V \to W \) is an isomorphism if it is an bijection. Two vector spaces are isomorphic (written as \( V \cong W \)) if there is an isomorphism \( T : V \to W \).
Definition

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We call a map \( f : X \rightarrow Y \) an **injection** (*injective, one-to-one, 1-1*) if \( f(x_1) \neq f(x_2) \) for all \( x_1 \neq x_2 \in X \).

We call a map \( f : X \rightarrow Y \) a **surjection** (*surjective, onto*) if \( f(X) = Y \), i.e., for every \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \).

We call a map \( f : X \rightarrow Y \) a **bijection** (*bijective*) if it is 1-1 and onto. A bijection \( f : X \rightarrow Y \) has an **inverse** \( f^{-1} : Y \rightarrow X \) such that \( f \circ f^{-1} = 1_Y \) and \( f^{-1} \circ f = 1_X \), where \( 1_X \) and \( 1_Y \) are the identity maps on \( X \) and \( Y \).

**Isomorphism**

A linear transformation \( T : V \rightarrow W \) is an **isomorphism** if it is an bijection. Two vector spaces are isomorphic (written as \( V \cong W \)) if there is an isomorphism \( T : V \rightarrow W \).
Definition of Injection, Surjection and Bijection

We call a map \( f : X \to Y \) an injection (injective, one-to-one, 1-1) if \( f(x_1) \neq f(x_2) \) for all \( x_1 \neq x_2 \in X \).

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Isomorphism

A linear transformation \( T : V \to W \) is an isomorphism if it is an bijection. Two vector spaces are isomorphic (written as \( V \cong W \)) if there is an isomorphism \( T : V \to W \).
Theorem

Let $T : V \to W$ be a linear transformation. Then

- $T$ is 1-1 if and only if $K(T) = \{0\}$;
- $T$ is onto if and only if $R(T) = W$.

Proof.

If $T$ is 1-1, then $T(v) \neq T(0) = 0$ for all $v \neq 0$. Therefore, $v \notin K(T)$ for all $v \neq 0$. That is, $K(T) = \{0\}$.

Suppose that $K(T) = \{0\}$. If $T$ is not 1-1, then there exists $v_1 \neq v_2$ such that $T(v_1) = T(v_2)$. Then

$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0$$

and $v_1 - v_2 \neq 0 \in K(T)$. Contradiction.
Theorem

Let $T : V \rightarrow W$ be a linear transformation. Then

- $T$ is 1-1 if and only if $\ker(T) = \{0\}$;
- $T$ is onto if and only if $\text{im}(T) = W$.

Proof.

If $T$ is 1-1, then $T(v) \neq T(0) = 0$ for all $v \neq 0$. Therefore, $v \notin \ker(T)$ for all $v \neq 0$. That is, $\ker(T) = \{0\}$.

Suppose that $\ker(T) = \{0\}$. If $T$ is not 1-1, then there exists $v_1 \neq v_2$ such that $T(v_1) = T(v_2)$. Then

$$T(v_1 - v_2) = T(v_1) - T(v_2) = 0$$

and $v_1 - v_2 \neq 0 \in \ker(T)$. Contradiction.
Injective and Surjective Linear Transformations

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Let \( T : V \rightarrow W \) be a linear transformation. Then

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Proof.

If \( T \) is 1-1, then \( T(\mathbf{v}) \neq T(0) = 0 \) for all \( \mathbf{v} \neq 0 \). Therefore, \( \mathbf{v} \notin K(T) \) for all \( \mathbf{v} \neq 0 \). That is, \( K(T) = \{0\} \).

Suppose that \( K(T) = \{0\} \). If \( T \) is not 1-1, then there exists \( \mathbf{v}_1 \neq \mathbf{v}_2 \) such that \( T(\mathbf{v}_1) = T(\mathbf{v}_2) \). Then

\[
T(\mathbf{v}_1 - \mathbf{v}_2) = T(\mathbf{v}_1) - T(\mathbf{v}_2) = 0
\]

and \( \mathbf{v}_1 - \mathbf{v}_2 \neq 0 \in K(T) \). Contradiction.
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- \( T \) is onto if and only if \( R(T) = W \).

Proof.

If \( T \) is 1-1, then \( T(\mathbf{v}) \neq T(0) = 0 \) for all \( \mathbf{v} \neq 0 \). Therefore, \( \mathbf{v} \notin K(T) \) for all \( \mathbf{v} \neq 0 \). That is, \( K(T) = \{0\} \).

Suppose that \( K(T) = \{0\} \). If \( T \) is not 1-1, then there exists \( \mathbf{v}_1 \neq \mathbf{v}_2 \) such that \( T(\mathbf{v}_1) = T(\mathbf{v}_2) \). Then

\[
T(\mathbf{v}_1 - \mathbf{v}_2) = T(\mathbf{v}_1) - T(\mathbf{v}_2) = 0
\]

and \( \mathbf{v}_1 - \mathbf{v}_2 \neq 0 \in K(T) \). Contradiction.
Inverse Linear Transformation

**Theorem**

Let $T: V \rightarrow W$ be an isomorphism. Then $T^{-1}$ is also a linear transformation.

In addition, if $V$ is finite dimensional, $\dim V = \dim W$ and

$$[T]_{B,C}^{-1} = [T^{-1}]_{C,B}.$$ 

$T^{-1}$ is a linear transformation.

For $w_1, w_2 \in W$, let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Then

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = w_1 + cw_2$$

$$\Rightarrow \quad \underbrace{v_1 + c}_{T^{-1}(w_1)} \underbrace{v_2}_{T^{-1}(w_2)} = T^{-1}(w_1 + cw_2)$$
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Inverse Linear Transformation

**Theorem**

Let \( T : V \rightarrow W \) be an isomorphism. Then \( T^{-1} \) is also a linear transformation.

In addition, if \( V \) is finite dimensional, \( \dim V = \dim W \) and

\[
[T]^{-1}_{B,C} = [T^{-1}]_{C,B}.
\]

\( T^{-1} \) is a linear transformation.

For \( w_1, w_2 \in W \), let \( v_1 = T^{-1}(w_1) \) and \( v_2 = T^{-1}(w_2) \). Then

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T(v_1 + cv_2) = T(v_1) + cT(v_2) = w_1 + cw_2
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\[\Rightarrow \quad \underbrace{v_1}_{T^{-1}(w_1)} + c \underbrace{v_2}_{T^{-1}(w_2)} = T^{-1}(w_1 + cw_2)\]
Proof.

If \( \dim V = n < \infty \), let \( B = \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \). Then

\[
R(T) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.
\]

Since \( T \) is onto, \( R(T) = W \) and hence \( \dim W = m \leq n \). Let \( C = \{w_1, w_2, \ldots, w_m\} \) be a basis of \( W \). Then

\[
R(T^{-1}) = \text{Span}\{T^{-1}(w_1), T^{-1}(w_2), \ldots, T^{-1}(w_m)\}.
\]

Since \( T^{-1} \) is onto, \( R(T^{-1}) = V \) and hence \( \dim V = n \leq m \). So \( m = n \). Finally,

\[
T \circ T^{-1} = 1_W \Rightarrow [T]_{B,C}[T^{-1}]_{C,B} = [1_W]_{C,C} = I
\]

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\Rightarrow [T]_{B,C}^{-1} = [T^{-1}]_{C,B}
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Proof.

If \( \dim V = n < \infty \), let \( B = \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \). Then

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\]

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Proof.

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