Outline

1. Linear Dependence

2. Basis and Dimension
Linear Dependence in \( \mathbb{R}[x] \)

Let \( f_1(x), f_2(x), \ldots, f_m(x) \) be \( m \) polynomials of degree \(< n\):

\[
f_i(x) = a_{i1} + a_{i2}x + \ldots + a_{in}x^{n-1}
\]

for \( i = 1, 2, \ldots, m \). Then \( c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0 \) if and only if

\[
\begin{align*}
a_{11}c_1 & + a_{21}c_2 + \ldots + a_{m1}c_m = 0 \\
a_{12}c_1 & + a_{22}c_2 + \ldots + a_{m2}c_m = 0 \\
& \vdots \quad + \quad \vdots \quad + \quad \ddots \quad + \quad \vdots \quad = \quad \vdots \\
a_{1n}c_1 & + a_{2n}c_2 + \ldots + a_{mn}c_m = 0
\end{align*}
\]

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
& \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_m
\end{bmatrix}
= 0
\]
Therefore, \( f_1(x), f_2(x), \ldots, f_m(x) \) are linearly independent if and only if

\[
\text{rank } \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix} = m.
\]

Roughly, we “identify” a polynomial

\[
b_1 + b_2x + \ldots + b_nx^{n-1} \leftrightarrow (b_1, b_2, \ldots, b_n).
\]

This is better explained later using linear transformation.
Choose $m$ distinct numbers $x_1, x_2, \ldots, x_m$ and let

$$B = \begin{bmatrix}
  f_1(x_1) & f_1(x_2) & \cdots & f_1(x_m) \\
  f_2(x_1) & f_2(x_2) & \cdots & f_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m(x_1) & f_m(x_2) & \cdots & f_m(x_m)
\end{bmatrix}$$

We claim that if $f_1, f_2, \ldots, f_m$ is linearly dependent, then $B$ is singular.
Linear Dependence in $\mathbb{R}[x]$

Therefore, $\text{rank}(A) \geq \text{rank}(B)$. If $\text{rank}(B) = m$, then $f_1, f_2, \ldots, f_m$ are linearly independent.

Caution: The converse fails.
If \( c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0 \), then

\[
c_1 f_1'(x) + c_2 f_2'(x) + \ldots + c_m f_m'(x) = 0
\]

More generally, differentiating \( k \) times, we obtain

\[
c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x) + \ldots + c_m f_m^{(k)}(x) = 0
\]

For \( k = 0, 1, \ldots, m - 1 \), we obtain

\[
\begin{bmatrix}
  f_1 & f_2 & \ldots & f_m \\
  f_1' & f_2' & \ldots & f_m' \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(m-1)} & f_2^{(m-1)} & \ldots & f_m^{(m-1)}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_m
\end{bmatrix} = 0
\]
Therefore, \( f_1(x), f_2(x), \ldots, f_m(x) \) are linearly independent if (and only if)

\[
W(f_1, f_2, \ldots, f_m) = \det \begin{bmatrix}
f_1 & f_2 & \cdots & f_m \\
f'_1 & f'_2 & \cdots & f'_m \\
\vdots & \vdots & \ddots & \vdots \\
f^{(m-1)}_1 & f^{(m-1)}_2 & \cdots & f^{(m-1)}_m \\
\end{bmatrix} \neq 0
\]

where \( W(f_1, f_2, \ldots, f_m) \) is called the Wronskian of \( f_1, f_2, \ldots, f_m \).

Note that \( W(f_1, f_2, \ldots, f_m)(x) \) is a function of \( x \) and \( W(f_1, f_2, \ldots, f_m)(x) \neq 0 \) if \( W(f_1, f_2, \ldots, f_m)(c) \neq 0 \) for some \( c \).
Example of Linear Dependence in $\mathbb{R}[x]$

Example. Show that $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ are linearly independent in $\mathbb{R}[x]$ for all $n \geq 0$.

Proof using coefficient matrix.

Since

\[
\begin{align*}
1 &= 1 \\
x + 1 &= 1 + x \\
(x + 1)^2 &= 1 + 2x + x^2 \\
\vdots \\
(x + 1)^n &= \binom{n}{0} + \binom{n}{1}x + \ldots + x^n
\end{align*}
\]
Linear Dependence
Basis and Dimension

Example of Linear Dependence in $\mathbb{R}[x]$

Cont.

The coefficient matrix

$$
\begin{bmatrix}
1 & 1 & \cdots & \cdots & 1 \\
1 & 2 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
(n_0) & (n_1) & \cdots & \cdots & 1
\end{bmatrix}
$$

is lower-triangular and with all diagonal entries 1. So it is nonsingular and $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ are linearly independent.
Example of Linear Dependence in $\mathbb{R}[x]$

Proof by Specialization.

Let $f_m(x) = (1 + x)^m$. Since

$$
\begin{bmatrix}
    f_0(0) & f_0(1) & \ldots & f_0(n) \\
    f_1(0) & f_1(1) & \ldots & f_1(n) \\
    f_2(0) & f_2(1) & \ldots & f_2(n) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_n(0) & f_n(1) & \ldots & f_n(n)
\end{bmatrix}
= 
\begin{bmatrix}
    1 & 1 & \ldots & 1 \\
    1 & 2 & \ldots & n+1 \\
    1^2 & 2^2 & \ldots & (n+1)^2 \\
    \vdots & \vdots & \ddots & \vdots \\
    1^n & 2^n & \ldots & (n+1)^n
\end{bmatrix}
$$

is nonsingular, $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ are linearly independent.
Example of Linear Dependence in $\mathbb{R}[x]$

Proof using Wronskian.

The Wronskian of $1, x+1, (x+1)^2, \ldots, (x+1)^n$ is

$$\begin{vmatrix} 1 & x+1 & (x+1)^2 & \ldots & (x+1)^n \\ 0 & 1! & * & \ldots & * \\ 0 & 0 & 2! & \ldots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & n! \end{vmatrix} = (1!)(2!)(n!) \neq 0.$$ 

Therefore, $1, x+1, (x+1)^2, \ldots, (x+1)^n$ are linearly independent.
Basis

Definition

We call a subset \( B \subset V \) a basis of vector space \( V \) if

- \( B \) is linearly independent;
- \( \text{Span}(B) = V \).

Remarks.

- In other words, a basis \( B \) is the smallest set of vectors spanning \( V \), i.e., \( \text{Span}(B) = V \) and \( \text{Span}(B') \neq V \) for all \( B' \subset B \) and \( B' \neq B \).
- Given a basis \( B \), every vector \( \mathbf{v} \in V \) can be written as a linear combination of vectors in \( B \) in a unique way:

\[
\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_n \mathbf{v}_n = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n
\]

\[
\Rightarrow (b_1 - c_1)\mathbf{v}_1 + (b_2 - c_2)\mathbf{v}_2 + \ldots + (b_n - c_n)\mathbf{v}_n = 0.
\]
Definition

If a vector space \( V \) has a finite basis \( B \), we call \( |B| \), the number of vectors in \( B \), the *dimension* of \( V \). We say that \( V \) is finite dimensional or has finite dimension of

\[
\dim V = |B|.
\]

Question. Is it possible that \( |B| \neq |B'| \) for two finite bases of \( V \)? Namely, is \( \dim V \) well-defined?

Theorem (\( \dim V \) is well-defined)

Let \( V \) be a vector space. If \( S = \{u_1, u_2, ..., u_m\} \) and \( T = \{v_1, v_2, ..., v_n\} \) are two finite bases of \( V \), then \( m = n \).
Proof.

Since \( \text{Span}(S) = V \), each \( \mathbf{v}_i \) is a linear combination of \( \mathbf{u}_j \):

\[
\mathbf{v}_i = a_{i1}\mathbf{u}_1 + a_{i2}\mathbf{u}_2 + \ldots + a_{im}\mathbf{u}_m
\]

\[
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\vdots \\
\mathbf{v}_n
\end{bmatrix} = 
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{11} & a_{12} & \ldots & a_{1m} \\
a_{n1} & a_{n2} & \ldots & a_{nm}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
\vdots \\
\mathbf{u}_n
\end{bmatrix}
\]

If \( n > m \), \( A^T \) is an \( m \times n \) matrix and hence \( A^T\mathbf{x} = 0 \) has a nonzero solution.
Cont.

That is, there exist \(c_1, c_2, \ldots, c_n\), not all zero, such that

\[
A^T \begin{bmatrix}
   c_1 \\
   c_2 \\
   \vdots \\
   c_n
\end{bmatrix} = 0 \iff \begin{bmatrix}
   c_1 & c_2 & \ldots & c_n
\end{bmatrix} A = 0
\]

\[
\Rightarrow \begin{bmatrix}
   c_1 & c_2 & \ldots & c_n
\end{bmatrix} \begin{bmatrix}
   v_1 \\
   v_2 \\
   \vdots \\
   v_n
\end{bmatrix} = \begin{bmatrix}
   c_1 & c_2 & \ldots & c_n
\end{bmatrix} A \begin{bmatrix}
   u_1 \\
   u_2 \\
   \vdots \\
   u_n
\end{bmatrix} = 0.
\]

So \(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0\) and \(T\) is linearly dependent.
Basic facts about basis and dimension

- If \( \dim V = n \), every set of \( n \) linearly independent vectors is a basis of \( V \).
- If \( \dim V = n \), every set of \( m > n \) vectors is linear dependent.
- Given a basis \( B = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) of \( V \) of \( \dim V = n \), every vector \( \mathbf{v} \) can be written as a linear combination of \( \mathbf{v}_i \) in a unique way:
  \[
  \mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_n \mathbf{v}_n
  \]
where \( (b_1, b_2, \ldots, b_n) \) are the coordinates of \( \mathbf{v} \) relative to (with respect to) the (ordered) basis \( B \).
Basic facts about basis and dimension

Theorem
Every set of \( m < n = \dim V \) linearly independent vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \) can be complemented to a basis of \( V \). That is, there exists \( \mathbf{v}_{m+1}, \ldots, \mathbf{v}_n \) such that \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) is a basis.

Proof.
Let \( W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \). Since \( m < n \), \( W \neq V \). So there exists \( \mathbf{v}_{m+1} \in V \setminus W \). Let \( W' = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}\} \). If \( W' = V \), we are done. Otherwise, we choose \( \mathbf{v}_{m+2} \in V \setminus W' \) and continue.
Examples of Bases and Dimensions

- $\mathbb{R}^n$ has dimension $n$ with the standard basis $\{e_1, e_2, \ldots, e_n\}$ where $e_k$ is the $k$-th row (or column) of the $n \times n$ identity matrix.
  
  More generally, the row (or column) vectors of an $n \times n$ nonsingular matrix are a basis of $\mathbb{R}^n$.

- $\mathbb{R}[x]$ is infinite dimensional with basis $\{1, x, x^2, \ldots, x^n, \ldots\}$. Also, a set $\{f_0(x), f_1(x), \ldots, f_n(x), \ldots\}$ is a basis if $\deg f_n(x) = n$.

- Let $\mathbb{R}_<n[x]$ be the set of polynomials in $\mathbb{R}[x]$ of degree $< n$. Then $\mathbb{R}_<n[x]$ has dimension $n$ with basis $\{1, x, \ldots, x^{n-1}\}$. Also, a set $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ is a basis if $\deg f_k(x) = k$ for $k = 0, 1, \ldots, n - 1$.

- Let $W = \{f \in \mathbb{R}[x] : \deg f < n, f(1) = 0\}$. Then $W$ has dimension $\dim W = n - 1$ with basis $\{x - 1, (x - 1)^2, \ldots, (x - 1)^{n-1}\}$. 

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Linear Algebra II Lecture 6
Let $W = \{ f \in \mathbb{R}[x] : \deg f < n, f(1) = f(2) \}$. Every $f(x) \in W$ can be written as
\[
f(x) = (x - 1)(x - 2)g(x) + b
\]
for some constant $b$ and $g(x) \in \mathbb{R}[x]$ of degree $< n - 2$. Then $W$ has dimension $\dim W = n - 1$ with basis
\[
\{ 1, (x - 1)(x - 2), (x - 1)(x - 2)x, \ldots, (x - 1)(x - 2)x^{n-3} \}.
\]
Let $W = \{ f \in \mathbb{R}[x] : \deg f < n, f''(1) = 0 \}$. Every $f \in \mathbb{R}[x]$ can be written the Taylor series at 1
\[
f(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \ldots + \frac{f^{(k)}(1)}{k!}(x - 1)^k + \ldots
\]
Therefore, $W$ has dimension $n - 1$ with basis
\[
\{ 1, (x - 1), (x - 1)^3, \ldots, (x - 1)^{n-1} \}.
\]