Outline

1. Linear Dependence
2. Basis and Dimension
Linear Dependence in $\mathbb{R}[x]$

Let $f_1(x), f_2(x), \ldots, f_m(x)$ be $m$ polynomials of degree $< n$:

$$f_i(x) = a_{i1} + a_{i2}x + \ldots + a_{in}x^{n-1}$$

for $i = 1, 2, \ldots, m$. Then $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0$ if and only if

$$
\begin{align*}
    a_{11} c_1 + a_{21} c_2 + \ldots + a_{m1} c_m &= 0 \\
    a_{12} c_1 + a_{22} c_2 + \ldots + a_{m2} c_m &= 0 \\
    \vdots &+ \vdots + \ldots + \vdots = \vdots \\
    a_{1n} c_1 + a_{2n} c_2 + \ldots + a_{mn} c_m &= 0
\end{align*}
$$

$$
\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}^T
\begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_m
\end{bmatrix} = 0
$$
Therefore, $f_1(x), f_2(x), \ldots, f_m(x)$ are linearly independent if and only if

$$\text{rank} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = m.$$ 

Roughly, we “identify” a polynomial

$$b_1 + b_2x + \ldots + b_nx^{n-1} \leftrightarrow (b_1, b_2, \ldots, b_n).$$

This is better explained later using linear transformation.
Therefore, $f_1(x)$, $f_2(x)$, ..., $f_m(x)$ are linearly independent if and only if

$$\text{rank} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = m.$$ 

Roughly, we “identify” a polynomial

$$b_1 + b_2x + \ldots + b_nx^{n-1} \leftrightarrow (b_1, b_2, \ldots, b_n).$$

This is better explained later using linear transformation.
Choose $m$ distinct numbers $x_1, x_2, \ldots, x_m$ and let

$$B = \begin{bmatrix}
  f_1(x_1) & f_1(x_2) & \ldots & f_1(x_m) \\
  f_2(x_1) & f_2(x_2) & \ldots & f_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m(x_1) & f_m(x_2) & \ldots & f_m(x_m)
\end{bmatrix}$$

We claim that if $f_1, f_2, \ldots, f_m$ is linearly dependent, then $B$ is singular.
Choose $m$ distinct numbers $x_1, x_2, \ldots, x_m$ and let

\[
B = \begin{bmatrix}
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  f_2(x_1) & f_2(x_2) & \ldots & f_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m(x_1) & f_m(x_2) & \ldots & f_m(x_m)
\end{bmatrix}
\]

We claim that if $f_1, f_2, \ldots, f_m$ is linearly dependent, then $B$ is singular.
Linear Dependence in $\mathbb{R}[x]$ 

$$
\begin{bmatrix}
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    f_2(x_1) & f_2(x_2) & \ldots & f_2(x_m) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_m(x_1) & f_m(x_2) & \ldots & f_m(x_m)
\end{bmatrix}
\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
= 
\begin{bmatrix}
    1 & 1 & \ldots & 1 \\
    x_1 & x_2 & \ldots & x_m \\
    \vdots & \vdots & \ddots & \vdots \\
    x_1^{n-1} & x_2^{n-1} & \ldots & x_m^{n-1}
\end{bmatrix}
$$

Therefore, $\text{rank}(A) \geq \text{rank}(B)$. If $\text{rank}(B) = m$, then $f_1, f_2, \ldots, f_m$ are linearly independent. Caution: The converse fails.
Linear Dependence in $\mathbb{R}[x]$

$$\begin{bmatrix}
  f_1(x_1) & f_1(x_2) & \ldots & f_1(x_m) \\
  f_2(x_1) & f_2(x_2) & \ldots & f_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_m(x_1) & f_m(x_2) & \ldots & f_m(x_m)
\end{bmatrix} = \begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix} \begin{bmatrix}
  1 & 1 & \ldots & 1 \\
  x_1 & x_2 & \ldots & x_m \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^{n-1} & x_2^{n-1} & \ldots & x_m^{n-1}
\end{bmatrix}$$

Therefore, $\text{rank}(A) \geq \text{rank}(B)$. If $\text{rank}(B) = m$, then $f_1, f_2, \ldots, f_m$ are linearly independent. Caution: The converse fails.
If $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0$, then

$$c_1 f_1'(x) + c_2 f_2'(x) + \ldots + c_m f_m'(x) = 0$$

More generally, differentiating $k$ times, we obtain

$$c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x) + \ldots + c_m f_m^{(k)}(x) = 0$$

For $k = 0, 1, \ldots, m - 1$, we obtain

$$\begin{bmatrix}
    f_1 & f_2 & \ldots & f_m \\
    f_1' & f_2' & \ldots & f_m' \\
    \vdots & \vdots & \ddots & \vdots \\
    f_1^{(m-1)} & f_2^{(m-1)} & \ldots & f_m^{(m-1)}
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_m
\end{bmatrix} = 0$$
If \( c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0 \), then

\[
c_1 f_1'(x) + c_2 f_2'(x) + \ldots + c_m f_m'(x) = 0
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More generally, differentiating \( k \) times, we obtain

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  f_1 & f_2 & \ldots & f_m \\
  f_1' & f_2' & \ldots & f_m' \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(m-1)} & f_2^{(m-1)} & \ldots & f_m^{(m-1)}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_m
\end{bmatrix} = 0
\]
Linear Dependence in $\mathbb{R}[x]$ by Wronskian

If $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x) = 0$, then

$$c_1 f_1'(x) + c_2 f_2'(x) + \ldots + c_m f_m'(x) = 0$$

More generally, differentiating $k$ times, we obtain

$$c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x) + \ldots + c_m f_m^{(k)}(x) = 0$$

For $k = 0, 1, \ldots, m - 1$, we obtain

$$\begin{bmatrix} f_1 & f_2 & \ldots & f_m \\ f_1' & f_2' & \ldots & f_m' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \ldots & f_m^{(m-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = 0$$
Therefore, \( f_1(x), f_2(x), \ldots, f_m(x) \) are linearly independent if (and only if)

\[
W(f_1, f_2, \ldots, f_m) = \det \begin{bmatrix}
  f_1 & f_2 & \cdots & f_m \\
  f'_1 & f'_2 & \cdots & f'_m \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)}
\end{bmatrix} \neq 0
\]

where \( W(f_1, f_2, \ldots, f_m) \) is called the \textit{Wronskian} of \( f_1, f_2, \ldots, f_m \).

Note that \( W(f_1, f_2, \ldots, f_m)(x) \) is a function of \( x \) and \( W(f_1, f_2, \ldots, f_m)(x) \neq 0 \) if \( W(f_1, f_2, \ldots, f_m)(c) \neq 0 \) for some \( c \).
Therefore, $f_1(x), f_2(x), \ldots, f_m(x)$ are linearly independent if (and only if)

$$W(f_1, f_2, \ldots, f_m) = \det \begin{bmatrix}
  f_1 & f_2 & \cdots & f_m \\
  f'_1 & f'_2 & \cdots & f'_m \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{(m-1)}^{(m-1)} & f_{2}^{(m-1)} & \cdots & f_{m}^{(m-1)}
\end{bmatrix} \neq 0$$

where $W(f_1, f_2, \ldots, f_m)$ is called the Wronskian of $f_1, f_2, \ldots, f_m$.

Note that $W(f_1, f_2, \ldots, f_m)(x)$ is a function of $x$ and $W(f_1, f_2, \ldots, f_m)(x) \neq 0$ if $W(f_1, f_2, \ldots, f_m)(c) \neq 0$ for some $c$. 
Example of Linear Dependence in $\mathbb{R}[x]$

Example. Show that $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ are linearly independent in $\mathbb{R}[x]$ for all $n \geq 0$.

Proof using coefficient matrix.

Since

$$1 = 1$$
$$x + 1 = 1 + x$$
$$(x + 1)^2 = 1 + 2x + x^2$$
$$\ldots$$
$$(x + 1)^n = \binom{n}{0} + \binom{n}{1}x + \ldots + x^n$$
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&\quad \ldots \\
(x + 1)^n &= \binom{n}{0} + \binom{n}{1}x + \ldots + x^n
\end{align*}
\]
The coefficient matrix

\[
\begin{bmatrix}
1 & 1 & & \\
1 & 1 & 2 & 1 \\
1 & 2 & 1 & \\
\vdots & \vdots & \vdots & \ddots \\
(0) & (1) & (n) & \ldots & \ldots & 1
\end{bmatrix}
\]

is lower-triangular and with all diagonal entries 1. So it is nonsingular and 1, \(x + 1\), \((x + 1)^2\), \ldots, \((x + 1)^n\) are linearly independent.
Example of Linear Dependence in $\mathbb{R}[x]$ 

Proof by Specialization.

Let $f_m(x) = (1 + x)^m$. Since

$$
\begin{bmatrix}
  f_0(0) & f_0(1) & \ldots & f_0(n) \\
  f_1(0) & f_1(1) & \ldots & f_1(n) \\
  f_2(0) & f_2(1) & \ldots & f_2(n) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_n(0) & f_n(1) & \ldots & f_n(n)
\end{bmatrix}
= 
\begin{bmatrix}
  1 & 1 & \ldots & 1 \\
  1 & 2 & \ldots & n + 1 \\
  1^2 & 2^2 & \ldots & (n + 1)^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  1^n & 2^n & \ldots & (n + 1)^n
\end{bmatrix}
$$

is nonsingular, $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ are linearly independent.
Example of Linear Dependence in $\mathbb{R}[x]$

**Proof using Wronskian.**

The Wronskian of $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ is

$$
\begin{vmatrix}
1 & x + 1 & (x + 1)^2 & \ldots & (x + 1)^n \\
0 & 1! & * & \ldots & * \\
0 & 0 & 2! & \ldots & * \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & n!
\end{vmatrix}
$$

$$
= (1!)(2!)(n!) \neq 0.
$$

Therefore, $1, x + 1, (x + 1)^2, \ldots, (x + 1)^n$ are linearly independent.
Basis

Definition

We call a subset $B \subset V$ a *basis* of vector space $V$ if

- $B$ is linearly independent;
- $\text{Span}(B) = V$.

Remarks.

In other words, a basis $B$ is the smallest set of vectors spanning $V$, i.e., $\text{Span}(B) = V$ and $\text{Span}(B') \neq V$ for all $B' \subset B$ and $B' \neq B$.

Given a basis $B$, every vector $\mathbf{v} \in V$ can be written as a linear combination of vectors in $B$ in a unique way:

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_n \mathbf{v}_n = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n$$

$$\Rightarrow (b_1 - c_1) \mathbf{v}_1 + (b_2 - c_2) \mathbf{v}_2 + \ldots + (b_n - c_n) \mathbf{v}_n = 0.$$
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\[
v = b_1v_1 + b_2v_2 + \ldots + b_nv_n = c_1v_1 + c_2v_2 + \ldots + c_nv_n
\]

\[
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- Given a basis \( B \), every vector \( \mathbf{v} \in V \) can be written as a linear combination of vectors in \( B \) in a unique way:

\[
\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_n \mathbf{v}_n = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n
\]

\[
\Rightarrow (b_1 - c_1) \mathbf{v}_1 + (b_2 - c_2) \mathbf{v}_2 + \ldots + (b_n - c_n) \mathbf{v}_n = 0.
\]
Definition

If a vector space $V$ has a finite basis $B$, we call $|B|$, the number of vectors in $B$, the *dimension* of $V$. We say that $V$ is finite dimensional or has finite dimension of

$$\dim V = |B|.$$ 

Question. Is it possible that $|B| \neq |B'|$ for two finite bases of $V$? Namely, is $\dim V$ well-defined?

Theorem (dim $V$ is well-defined)

Let $V$ be a vector space. If $S = \{u_1, u_2, ..., u_m\}$ and $T = \{v_1, v_2, ..., v_n\}$ are two finite bases of $V$, then $m = n$. 

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Linear Algebra II Lecture 6
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Linear Algebra II Lecture 6
Definition of \( \text{dim } V \)

**Proof.**

Since \( \text{Span}(S) = V \), each \( \mathbf{v}_i \) is a linear combination of \( \mathbf{u}_j \):

\[
\mathbf{v}_i = a_{i1}\mathbf{u}_1 + a_{i2}\mathbf{u}_2 + ... + a_{im}\mathbf{u}_m
\]

\[
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
... \\
\mathbf{v}_n
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & ... & a_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & ... & a_{nm}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
... \\
\mathbf{u}_n
\end{bmatrix}
\]

If \( n > m \), \( A^T \) is an \( m \times n \) matrix and hence \( A^T \mathbf{x} = 0 \) has a nonzero solution.
That is, there exist $c_1, c_2, \ldots, c_n$, not all zero, such that

$$A^T \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0 \iff \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} A = 0$$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = 0.$$ 

So $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ and $T$ is linearly dependent. Contradiction. So $n \leq m$. Similarly, $m \leq n$. So $m = n$. 
Continued.

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So $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ and $T$ is linearly dependent. Contradiction. So $n \leq m$. Similarly, $m \leq n$. So $m = n$. 
Basic facts about basis and dimension

- If $\dim V = n$, every set of $n$ linearly independent vectors is a basis of $V$.
- If $\dim V = n$, every set of $m > n$ vectors is linear dependent.
- Given a basis $B = \{v_1, v_2, ..., v_n\}$ of $V$ of $\dim V = n$, every vector $v$ can be written as a linear combination of $v_i$ in a unique way:
  \[ v = b_1 v_1 + b_2 v_2 + ... + b_n v_n \]
  where $(b_1, b_2, ..., b_n)$ are the coordinates of $v$ relative to (with respect to) the (ordered) basis $B$. 

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Linear Algebra II Lecture 6
Basic facts about basis and dimension

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Basic facts about basis and dimension

Theorem
Every set of $m < n = \dim V$ linearly independent vectors $v_1, v_2, \ldots, v_m$ can be complemented to a basis of $V$. That is, there exists $v_{m+1}, \ldots, v_n$ such that $v_1, v_2, \ldots, v_n$ is a basis.

Proof.
Let $W = \text{Span}\{v_1, v_2, \ldots, v_m\}$. Since $m < n$, $W \neq V$. So there exists $v_{m+1} \in V \setminus W$. Let $W' = \text{Span}\{v_1, v_2, \ldots, v_m, v_{m+1}\}$. If $W' = V$, we are done. Otherwise, we choose $v_{m+2} \in V \setminus W'$ and continue.
Basic facts about basis and dimension

**Theorem**
Every set of $m < n = \dim V$ linearly independent vectors $v_1, v_2, ..., v_m$ can be complemented to a basis of $V$. That is, there exists $v_{m+1}, ..., v_n$ such that $v_1, v_2, ..., v_n$ is a basis.

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Examples of Bases and Dimensions

- \( \mathbb{R}^n \) has dimension \( n \) with the standard basis \( \{ e_1, e_2, \ldots, e_n \} \) where \( e_k \) is the \( k \)-th row (or column) of the \( n \times n \) identity matrix. More generally, the row (or column) vectors of an \( n \times n \) nonsingular matrix are a basis of \( \mathbb{R}^n \).

- \( \mathbb{R}[x] \) is infinite dimensional with basis \( \{ 1, x, x^2, \ldots, x^n, \ldots \} \). Also, a set \( \{ f_0(x), f_1(x), \ldots, f_n(x), \ldots \} \) is a basis if \( \deg f_n(x) = n \).

- Let \( \mathbb{R}_{<n}[x] \) be the set of polynomials in \( \mathbb{R}[x] \) of degree < \( n \). Then \( \mathbb{R}_{<n}[x] \) has dimension \( n \) with basis \( \{ 1, x, \ldots, x^{n-1} \} \). Also, a set \( \{ f_0(x), f_1(x), \ldots, f_{n-1}(x) \} \) is a basis if \( \deg f_k(x) = k \) for \( k = 0, 1, \ldots, n - 1 \).

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Xi Chen  
Linear Algebra II Lecture 6
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\[
f(x) = (x - 1)(x - 2)g(x) + b
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for some constant \( b \) and \( g(x) \in \mathbb{R}[x] \) of degree \( < n - 2 \).

Then \( W \) has dimension \( \dim W = n - 1 \) with basis
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