Outline

1. Subspace
2. Span
Definition of Subspace

Definition. A subset $W \subset V$ of a vector space $V$ over $\mathbb{R}$ is a subspace if $W \neq \emptyset$ and

- $w_1 + cw_2 \in W$ for all $w_1, w_2 \in W$ and all $c \in \mathbb{R}$.

Note that

- a subspace $W \subset V$ is itself a vector space;
- a subspace $W \subset V$ is never empty ($0 \in W$);
- $\{0\}$ and $V$ are subspaces of $V$. 
Intersections of Subspaces

**Theorem**

The intersection $V_1 \cap V_2$ of two subspaces $V_1$ and $V_2$ of $V$ is also a subspace.

**Proof.**

For all $u, v \in V_1 \cap V_2$ and $c \in \mathbb{R}$,

\[
\begin{align*}
V_1 \text{ is a subspace} & \Rightarrow u + cv \in V_1 \\
V_2 \text{ is a subspace} & \Rightarrow u + cv \in V_2
\end{align*}
\]

\[\Rightarrow u + cv \in V_1 \cap V_2.\]

So $V_1 \cap V_2$ is a subspace of $V$. \qed
Examples of intersections of subspaces

Let $A_1$ and $A_2$ be two matrices of size $m_1 \times n$ and $m_2 \times n$, respectively. Let $N_{A_1} = \text{Nul}(A_1) = \{x : A_1x = 0\}$ and $N_{A_2} = \text{Nul}(A_2) = \{x : A_2x = 0\}$. Then

$$N_{A_1} \cap N_{A_2} = \left\{x : \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x = 0 \right\}$$

e.g.,

$$\{(x, y) : 3x + 4y = 0\} \cap \{(x, y) : x + y = 0\} = \{(x, y) : 3x + 4y = x + y = 0\} = \{(0, 0)\}$$

$$\{(x, y, z) : x - y + z = 0\} \cap \{(x, y, z) : 3x + 4y + 5z = 0\} = \{(x, y, z) : x - y + z = 3x + 4y + 5z = 0\} = \left\{(x, y, z) : \begin{bmatrix} 1 & -1 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \{(-9t, -2t, 7t)\}$$
More examples of intersections

Let \( F(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \} \), \( E = \{ f \in F(\mathbb{R}) : f(−x) \equiv f(x) \} \) and \( O = \{ f \in F(\mathbb{R}) : f(−x) \equiv −f(x) \} \). Both \( E \) and \( O \) are subspaces of \( F(\mathbb{R}) \). Their intersection is

\[
E \cap O = \{ 0 \}.
\]

If \( f(x) \in E \cap O \), then \( f(−x) = f(x) \) and \( f(−x) = −f(x) \) for all \( x \Rightarrow f(x) \equiv −f(x) \), i.e., \( f(x) \equiv 0 \).

Let \( U = \{ A \in M_{n \times n}(\mathbb{R}) : A = A^T \} \) and \( V = \{ A \in M_{n \times n}(\mathbb{R}) : −A = A^T \} \). Then

\[
U \cap V = \{ 0 \}
\]

since \( A = A^T \) and \( −A = A^T \Rightarrow A = −A \Rightarrow A = 0 \).
Subspaces spanned by subsets

**Theorem**

The union $V_1 \cup V_2$ of two subspaces of $V$ is not a subspace unless $V_1 \subset V_2$ or $V_2 \subset V_1$.

For example, let $V_1 = \{(x, y) : y = 0\}$ and $V_2 = \{(x, y) : x = 0\}$ and $V_1 \cup V_2$ is not a subspace of $\mathbb{R}^2$ since $v_1 = (1, 0) \in V_1$ and $v_2 = (0, 1) \in V_2$ but $v_1 + v_2 \not\in V_1 \cup V_2$. However, $\text{Span}(V_1 \cup V_2)$ is a subspace.

**Definition**

Let $S$ be a subset of $V$. Then the span $\text{Span}(S)$ of $S$ is the set consisting of

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$

for all $v_1, v_2, \ldots, v_n \in S$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$, i.e., all linear combinations of vectors in $S$. Also set $\text{Span}(\emptyset) = \{0\}$. 
Theorem

Span(S) is the smallest subspace of $V$ containing $S$. That is, if $W$ is a subspace of $V$ and $S \subset W$, then $\text{Span}(S) \subset W$.

Proof that $\text{Span}(S)$ is a subspace.

For $u, v \in \text{Span}(S)$, let

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n$$

and

$$v = b_1v_1 + b_2v_2 + \ldots + b_mv_m$$

for $a_i, b_j \in \mathbb{R}$ and $u_i, v_j \in S$. Then

$$u + cv = a_1u_1 + a_2u_2 + \ldots + a_nu_n$$

$$+ cb_1v_1 + cb_2v_2 + \ldots + cb_mv_m \in \text{Span}(S).$$
Proof that \( \text{Span}(S) \) is the smallest subspace containing \( S \).

Let \( W \subset V \) be a subspace with \( S \subset W \). Then

\[
a_1 u_1 + a_2 u_2 + \ldots + a_n u_n \in W
\]

for all \( a_1, a_2, \ldots, a_n \in \mathbb{R} \) and \( u_1, u_2, \ldots, u_n \in S \) since \( u_1, u_2, \ldots, u_n \in W \) and \( W \) is a subspace. So \( \text{Span}(S) \subset W \).

Or equivalently,

\[
\text{Span}(S) = \bigcap_{S \subset W} W
\]

Where \( W \subset V \) is a subspace.
**Definition**

The sum of two subspaces $V_1$ and $V_2$ of $V$ is

$$V_1 + V_2 = \{ v_1 + v_2 : v_1 \in V_1, v_2 \in V_2 \}$$

**Theorem**

Let $V_1$ and $V_2$ be two subspaces of $V$. Then

$$V_1 + V_2 = \text{Span}(V_1 \cup V_2).$$

For example, let $V_1 = \{(x, y) : y = 0\}$ and $V_2 = \{(x, y) : x = 0\}$. Then

$$\text{Span}(V_1 \cup V_2) = V_1 + V_2 = \mathbb{R}^2.$$
Examples of \( \text{Span}(S) \)

- Let \( S = \{1, x, x^2, \ldots, x^n\} \subset \mathbb{R}[x] \). Then
  \[
  \text{Span}(S) = \{a_0 + a_1 x + \ldots + a_n x^n\} = \{f(x) : \deg f(x) \leq n\}
  \]

- Let \( S = \{x^2 + y^2 = 1\} \). Then \((1, 0) \in S\) and \((0, 1) \in S\) so
  \[
  \text{Span}(S) \supset \text{Span}\{(1, 0), (0, 1)\} = \mathbb{R}^2
  \]
  and hence \( \text{Span}(S) = \mathbb{R}^2 \). 

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Let $V_0 = \{ f(x) | f(0) = 0 \}$ and $V_1 = \{ f(x) | f(1) = 0 \}$ be subspaces of $F(\mathbb{R})$. Let

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

Then for every $g(x) \in F(\mathbb{R})$,

$$g(x) = g(1)h(x) + (g(x) - g(1)h(x))$$

$$\in V_0 + (g(x) - g(1)h(x)) \in V_1$$

Therefore,

$$\text{Span}(V_0 \cup V_1) = V_0 + V_1 = F(\mathbb{R}).$$
Subspace Span

**Span(S) for \( S \subset \mathbb{R}^n \)**

Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \) be \( m \) vectors in \( \mathbb{R}^n \), represented by row vectors. Let \( A \) be the \( m \times n \) matrix

\[
A = \begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\vdots \\
\mathbf{v}_m
\end{bmatrix}.
\]

Then the *row space*

\[
\text{Row}(A) = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)
\]

the subspace of \( \mathbb{R}^n \) spanned by the rows of \( A \). Similarly, the column space \( \text{Col}(A) \) is the subspace spanned by the columns of \( A \). Clearly, \( \text{Row}(A) = \text{Col}(A^T) \).
Theorem

Let $A$ be an $m \times n$ matrix. Then

$$\text{Row}(A) = \text{Row}(BA)$$

for all nonsingular $m \times m$ matrices $B$.

Proof.

Let

$$A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}.$$

Then
Cont.

\[ BA = \begin{bmatrix}
  b_{11}v_1 + b_{12}v_2 + \ldots + b_{1m}v_m \\
  b_{21}v_1 + b_{22}v_2 + \ldots + b_{2m}v_m \\
  \vdots \\
  b_{m1}v_1 + b_{m2}v_2 + \ldots + b_{mm}v_m
\end{bmatrix}. \]

Since \( b_{i1}v_1 + b_{i2}v_2 + \ldots + b_{im}v_m \in \text{Span}(v_1, v_2, \ldots, v_m) \),

\[ \text{Row}(BA) \subset \text{Row}(A) \text{ for all } A \text{ and } B. \]

Let \( A' = BA \). Since \( B \) is nonsingular, \( A = B^{-1}A' \). So

\[ \text{Row}(B^{-1}A') \subset \text{Row}(A') \iff \text{Row}(A) \subset \text{Row}(BA). \]

We conclude that \( \text{Row}(A) = \text{Row}(BA) \).
Remarks on $\text{Row}(A) = \text{Row}(BA)$

- Similarly, $\text{Col}(AB) = \text{Col}(A)$ for all $n \times n$ nonsingular matrices $B$.
- Let $A'$ be a matrix in row echelon form obtained from $A$ by row reduction. Then $\text{Row}(A) = \text{Row}(A')$, e.g.,
  
  \[
  \begin{bmatrix}
  1 & 2 & 3 \\
  2 & 3 & 1 \\
  3 & 1 & 2
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{bmatrix}
  = \mathbb{R}^3
  \]

- $\text{Row}(A) = \mathbb{R}^n$ if and only if $\text{rank}(A) = n$, i.e., $A$ has full rank.
- If $A \in M_{n \times n}(\mathbb{R})$, $A$ is nonsingular if and only if $\text{Row}(A) = \mathbb{R}^n$ ($\text{Col}(A) = \mathbb{R}^n$), i.e., the row (column) vectors of $A$ span $\mathbb{R}^n$. 