Math 225 Assignment #5
Due Oct. 20, 2014

We use the following notations:
- $M_{m \times n}(\mathbb{R})$ is the set of all $m \times n$ matrices with real entries.
- $\mathbb{R}[x]$ is the set of all polynomials in $x$ with real coefficients.
- $F(D)$ is the set of all functions $f : D \rightarrow \mathbb{R}$.
- $\text{Span}(S)$ is the subspace spanned by the vectors in $S$.
- $\text{Row}(A)$ is the subspace spanned by the row vectors of $A$.
- $\text{Col}(A)$ is the subspace spanned by the column vectors of $A$.
- $\text{Nul}(A)$ is the null space of $A$ given by $\{x : Ax = 0\}$.
- $K(T)$ is the kernel of the linear transformation $T$.
- $R(T)$ is the range of the linear transformation $T$.
- $\text{rank}(T)$ is the rank of the linear transformation $T$.

(1) Let

$$A = \begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}$$

(a) Find a basis and the dimension of $\text{Nul}(A)$.
(b) Find a basis and the dimension of $\text{Col}(A)$.
(c) Let $T : M_{4 \times 4}(\mathbb{R}) \rightarrow M_{3 \times 4}(\mathbb{R})$ be the map given by

$$T(B) = AB.$$ 

Show that $T$ is a linear transformation.
(d) Let $T$ be the linear transformation given in (c). Find a basis and the dimension of $K(T)$.
(e) Let $T$ be the linear transformation given in (c). Find $\text{rank}(T)$ and a basis of $R(T)$.

(2) Which of the following statements are true and which are false? Justify your answer.

(a) If $\{u, v, w\}$ is a basis of a vector space $V$, then

$$\{u + v, v + w, w + u\}$$

is also a basis of $V$.
(b) Let $T : V \rightarrow W$ be a linear transformation. If $\{v_1, v_2, ..., v_n\}$ is a basis of $V$, then $\{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis of $W$. 

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(3) Let \( B_1 = \{(1,1), (1,-1)\} \) and \( B_2 = \{(2,3), (3,4)\} \) be two ordered basis of \( \mathbb{R}^2 \).

(a) Find the change-of-basis matrices \( P_{B_1 \leftarrow B_2} \) and \( P_{B_2 \leftarrow B_1} \).

(b) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map defined by

\[
T(v) = [v]_{B_1} + [v]_{B_2}.
\]

Show that \( T \) is a linear transformation.

(c) Find the matrix \( [T]_{B \leftarrow B} \) representing \( T \) under the standard basis \( B = \{e_1, e_2\} \).

(4) Let \( V = \{f(x) \in \mathbb{R}[x] : \deg f \leq 2\} \) be the vector space of real polynomials in \( x \) of degree \( \leq 2 \) and let \( T_1 : V \to V \) and \( T_2 : V \to V \) be two linear transformations given by

\[
T_1(f(x)) = f(x + 2) \quad \text{and} \quad T_2(f(x)) = f(x - 2).
\]

(a) Show that \( T_1 = T_2^{-1} \).

(b) Find the matrix representations \( [T_1]_{B \leftarrow B} \) and \( [T_2]_{B \leftarrow B} \) of \( T_1 \) and \( T_2 \), respectively, under the basis \( B = \{1, x, x^2\} \).

(c) Verify that

\[
[T_1]_{B \leftarrow B} = [T_2]_{B \leftarrow B}^{-1}.
\]