Solutions for Math 225 Assignment #2

(1) Determine whether $W$ is a subspace of $V$ and justify your answer:

(a) $V = \mathbb{R}^3$, $W = \{(a, 0, a) : a \in \mathbb{R}\}$.

Proof. Yes. For $a = 0$, $(a, 0, a) = (0, 0, 0) \in W$. For all $(a_1, 0, a_1), (a_2, 0, a_2) \in W$ and $c \in \mathbb{R}$, $(a_1, 0, a_1) + c(a_2, 0, a_2) = (a_1 + ca_2, 0, a_1 + ca_2) \in W$.

Therefore, $W$ is a subspace of $V$. □

(b) $V = \mathbb{R}^3$, $W = \{(a, b, |a|) : a, b \in \mathbb{R}\}$.

Proof. No, $W$ is not a subspace of $V$ since $(1, 0, 1) \in W$ but $(-1)(1, 0, 1) = (-1, 0, -1) \notin W$. □

(c) $V = M_{n \times n}(\mathbb{R})$, $W = \{\text{singular } n \times n \text{ matrices} \}$ for $n > 1$.

Proof. No, $W$ is not a subspace of $V$ since

$$
A = \begin{bmatrix}
1 & 0 \\
0 & \ddots \\
& & 0
\end{bmatrix}
$$

and $B = \begin{bmatrix}
0 & 1 \\
& & 1
\end{bmatrix} \in W$

but $A + B = I \notin W$. □

(d) $V = \mathbb{R}[x]$, $W = \{f(x^2 + 1) : f(x) \in \mathbb{R}[x]\}$.

Proof. Yes. For $f(x) \equiv 0$, $f(x^2 + 1) \equiv 0 \in W$. For all $f_1(x^2 + 1), f_2(x^2 + 1) \in W$ and $c \in \mathbb{R}$, $f_1(x^2 + 1) + cf_2(x^2 + 1) = (f_1 + cf_2)(x^2 + 1) \in W$.

So $W$ is a subspace of $V$. □

(2) For each of the following $3 \times n$ matrices $A$, determine whether $\text{Col}(A) = \mathbb{R}^3$:

\[
a) \begin{bmatrix}
1 & 2 \\
5 & 4 \\
1 & 1
\end{bmatrix} \quad b) \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} \quad c) \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]

\footnote{http://www.math.ualberta.ca/~xichen/math22514f/hw2sol.pdf}
Solution. For a $m \times 3$ matrix $A$, $\text{Col}(A) = \mathbb{R}^3$ if and only if $\text{rank}(A) = 3$. Find the rank of these matrices and we conclude that a) no b) yes c) no.

(3) Which of the following statements are true and which are false? Justify your answer.

(a) A system $Ax = b$ of linear equations has at least one solution if and only if $b \in \text{Col}(A)$.

Proof. True. Let $A = [v_1 \ v_2 \ ... \ v_n]$, where $v_1, v_2, ..., v_n$ are the column vectors of $A$.

If $Ax = b$ has a solution, let

$$x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

be a solution. Then

$$b = Ax = [v_1 \ v_2 \ ... \ v_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 v_1 + c_2 v_2 + ... + c_n v_n.$$ 

So $b \in \text{Span}\{v_1, v_2, ..., v_n\} = \text{Col}(A)$.

If $b \in \text{Col}(A)$, then

$$b = c_1 v_1 + c_2 v_2 + ... + c_n v_n = [v_1 \ v_2 \ ... \ v_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

for some $c_1, c_2, ..., c_n \in \mathbb{R}$. Therefore,

$$x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is a solution of $Ax = b$ and hence $Ax = b$ has a solution.

(b) Let $S_1$ and $S_2$ be two subsets of a vector space $V$. If $\text{Span}(S_1) \subset \text{Span}(S_2)$, then $S_1 \subset S_2$. 
Proof. False. Let $V = \mathbb{R}$, $S_1 = \{1\}$ and $S_2 = \{-1\}$. Then $\text{Span}(S_1) = \text{Span}(S_2) = V$. So $\text{Span}(S_1) \subset \text{Span}(S_2)$ but $S_1 \not\subset S_2$. \qed \\

(c) Let $S_1$ and $S_2$ be two subsets of a vector space $V$. If $S_1 \subset S_2$, then $\text{Span}(S_1) \subset \text{Span}(S_2)$.

Proof. True. For every $v \in \text{Span}(S_1),$

$$v = c_1v_1 + c_2v_2 + ... + c_nv_n$$

for some $c_1, c_2, ..., c_n \in \mathbb{R}$ and $v_1, v_2, ..., v_n \in S_1$. And since $S_1 \subset S_2$, $v_1, v_2, ..., v_n \in S_2$. Therefore, $v \in \text{Span}(S_2)$ and hence $\text{Span}(S_1) \subset \text{Span}(S_2)$. \qed \\

(d) $\text{Nul}(A) \subset \text{Nul}(BA)$ for all $m \times n$ matrices $A$ and $l \times m$ matrices $B$.

Proof. True. For $x \in \text{Nul}(A)$, $Ax = 0 \Rightarrow BAx = 0$ and hence $x \in \text{Nul}(BA)$. \qed \\

(4) Prove the following:
(a) Every square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.
(b) Let $W$ be the vector space of $n \times n$ matrices and let $U$ and $V$ be the subspaces of $W$ consisting of symmetric and skew-symmetric matrices, respectively. Then

$$W = U + V.$$ 

Proof. Let $A$ be a square matrix and let

$$B = \frac{1}{2}(A + A^T) \text{ and } C = \frac{1}{2}(A - A^T).$$ 

Then $A = B + C$.

Since

$$B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = B$$

$B$ is symmetric. And since

$$C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = -\frac{1}{2}(A - A^T) = -C$$

$C$ is skew-symmetric. So every square matrix is the sum of a symmetric matrix and a skew-symmetric matrix.
For every $A \in W$, $A = B + C$ where $B$ is symmetric and $C$ is skew-symmetric, i.e., $B \in U$ and $C \in V$. Therefore, $W = U + V$. □

(5) Find $V_1 \cap V_2$ and $V_1 + V_2$ for subspaces $V_1, V_2$ of $V$:

(a) $V = \mathbb{R}^3$, $V_1 = \{(x, y, z) : x + y = 2y + z = 0\}$ and $V_2 = \{(x, y, z) : x - y = 2y - z = 0\}$;

Proof. The intersection is $V_1 \cap V_2 = \{(x, y, z) : x + y = 2y + z = x - y = 2y - z = 0\} = \{(0, 0, 0)\}$.

Solving $x + y = 2y + z = 0$, we obtain $V_1 = \text{Span}\{(1, -1, 2)\}$.

Solving $x - y = 2y - z = 0$, we obtain $V_2 = \text{Span}\{(1, 1, 2)\}$.

Therefore, $V_1 + V_2 = \text{Span}\{(1, -1, 2), (1, 1, 2)\}$. □

(b) $V = \mathbb{R}[x]$, $V_1 = \{f(x) \in \mathbb{R}[x] : f(1) = 0\}$ and $V_2 = \{f(x) \in \mathbb{R}[x] : f(2) = 0\}$;

Solution. The intersection is $V_1 \cap V_2 = \{f(x) \in \mathbb{R}[x] : f(1) = f(2) = 0\}$

$= \{(x - 1)(x - 2)g(x) : g(x) \in \mathbb{R}[x]\}$.

Every polynomial $f(x) \in \mathbb{R}[x]$ can be written as $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = f(x) + f(1)(x - 2)$ and $f_2(x) = -f(1)(x - 2)$.

Since $f_1(1) = f(1) + f(1)(1 - 2) = 0$, $f_1(x) \in V_1$. Since $f_2(2) = -f(1)(2 - 2) = 0$, $f_2(x) \in V_2$. Therefore, $V_1 + V_2 = V$. □

(c) $V = M_{n \times n}(\mathbb{R})$, $V_1 = \{[a_{ij}]_{n \times n} : a_{ij} = 0 \text{ for all } i > j\}$ and $V_2 = \{[a_{ij}]_{n \times n} : a_{ij} = 0 \text{ for all } i < j\}$ (i.e. $V_1$ is the set of all upper triangular matrices and $V_2$ is the set of all lower triangular matrices).
Proof. The intersection is
\[ V_1 \cap V_2 = \{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ for } i \neq j \} \]
\[ = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} : b_1, b_2, ..., b_n \in \mathbb{R} \right\} . \]
That is, \( V_1 \cap V_2 \) is the subspace of diagonal matrices. Every \( n \times n \) matrix \( [a_{ij}]_{n \times n} \) can be written as
\[ [a_{ij}]_{n \times n} = [b_{ij}]_{n \times n} + [c_{ij}]_{n \times n} \]
where
\[ b_{ij} = \begin{cases} a_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases} \quad \text{and} \quad c_{ij} = \begin{cases} 0 & \text{if } i \leq j \\ a_{ij} & \text{if } i > j \end{cases} . \]
Since \( [b_{ij}] \in V_1 \) and \( [c_{ij}] \in V_2 \), \( V_1 + V_2 = M_{n \times n}(\mathbb{R}) \). □

(6) Let \( V_1 \) and \( V_2 \) be two subspaces of a vector space \( V \) satisfying \( V_1 \cap V_2 = \{0\} \). Show that if
\[ \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_2 \]
for \( \mathbf{v}_1, \mathbf{w}_1 \in V_1 \) and \( \mathbf{v}_2, \mathbf{w}_2 \in V_2 \), then
\[ \mathbf{v}_1 = \mathbf{w}_1 \text{ and } \mathbf{v}_2 = \mathbf{w}_2. \]

Proof. Since \( \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_2 \),
\[ \mathbf{v}_1 - \mathbf{w}_1 = \mathbf{w}_2 - \mathbf{v}_2 \]
Since \( \mathbf{v}_1, \mathbf{w}_1 \in W_1 \) and \( W_1 \) is a subspace, \( \mathbf{v}_1 - \mathbf{w}_1 \in W_1 \). Since \( \mathbf{v}_2, \mathbf{w}_2 \in W_2 \) and \( W_2 \) is a subspace, \( \mathbf{v}_2 - \mathbf{w}_2 \in W_2 \). Therefore,
\[ \mathbf{v}_1 - \mathbf{w}_1 = \mathbf{w}_2 - \mathbf{v}_2 \in W_1 \cap W_2. \]
And since \( W_1 \cap W_2 = \{0\} \),
\[ \mathbf{v}_1 - \mathbf{w}_1 = \mathbf{w}_2 - \mathbf{v}_2 = 0, \]
i.e., \( \mathbf{v}_1 = \mathbf{w}_1 \) and \( \mathbf{v}_2 = \mathbf{w}_2. \) □

(7) Let \( A \) be a square matrix. Show that if \( \text{Col}(A) = \text{Col}(A^2) \), then \( \text{Col}(A^3) = \text{Col}(A^3) \).
Proof. It is enough to prove that
\[ \text{Col}(AB) = \text{Col}(AC) \] if \( \text{Col}(B) = \text{Col}(C) \).

Applying the above to \( B = A \) and \( C = A^2 \), we conclude that
\[ \text{Col}(A^2) = \text{Col}(A^3) \] if \( \text{Col}(A) = \text{Col}(A^2) \).

Let \( B = [u_1 \ u_2 \ \ldots \ u_m] \) and \( C = [v_1 \ v_2 \ \ldots \ v_n] \). Then
\[ \text{Col}(B) = \text{Span}\{u_1, u_2, \ldots, u_m\}, \]
\[ \text{Col}(C) = \text{Span}\{v_1, v_2, \ldots, v_n\}, \]
\[ \text{Col}(AB) = \text{Col}(A [u_1 \ u_2 \ \ldots \ u_m]) = \text{Span}\{Au_1, Au_2, \ldots, Au_m\}, \]
\[ \text{Col}(AC) = \text{Col}(A [v_1 \ v_2 \ \ldots \ v_n]) = \text{Span}\{Av_1, Av_2, \ldots, Av_n\}. \]

Since \( \text{Col}(B) = \text{Col}(C) \),
\[ \text{Span}\{u_1, u_2, \ldots, u_m\} = \text{Span}\{v_1, v_2, \ldots, v_n\}. \]

Therefore,
\[ u_i \in \text{Span}\{v_1, v_2, \ldots, v_n\} \] for \( i = 1, 2, \ldots, m \) and
\[ v_j \in \text{Span}\{u_1, u_2, \ldots, u_m\} \] for \( j = 1, 2, \ldots, n \).

It follows that
\[ Au_i \in \text{Span}\{Av_1, Av_2, \ldots, Av_n\} \] for \( i = 1, 2, \ldots, m \) and
\[ Av_j \in \text{Span}\{Au_1, Au_2, \ldots, Au_m\} \] for \( j = 1, 2, \ldots, n \).

Consequently,
\[ \text{Span}\{Au_1, Au_2, \ldots, Au_m\} \subseteq \text{Span}\{Av_1, Av_2, \ldots, Av_n\} \] and
\[ \text{Span}\{Av_1, Av_2, \ldots, Av_n\} \subseteq \text{Span}\{Au_1, Au_2, \ldots, Au_m\}. \]

That is, \( \text{Col}(AB) \subseteq \text{Col}(AC) \) and \( \text{Col}(AC) \subseteq \text{Col}(AB) \). So \( \text{Col}(AB) = \text{Col}(AC) \). \( \square \)

(8) Let \( W \) be a subspace of a vector space \( V \) over \( \mathbb{R} \) and \( v \) be a vector in \( V \). Show that \( W \cup \{v\} \) is a subspace of \( V \) if and only if \( v \in W \).

Proof. If \( v \in W \), \( W \cup \{v\} = W \) is a subspace of \( V \).

Suppose that \( W' = W \cup \{v\} \) is a subspace. Since \( v \in W' \) and \( W' \) is a subspace,
\[ 2v \in W' = W \cup \{v\}. \]

Therefore, \( 2v \in W \) or \( 2v \in \{v\} \).

If \( 2v \in \{v\} \), then \( 2v = v \) and hence \( v = 0 \in W \).

If \( 2v \in W \), \( v = \frac{1}{2}(2v) \in W \) since \( W \) is a subspace. \( \square \)