Solutions for Math 225 Assignment #10

(1) Use Gram-Schmidt to find an orthogonal basis of each of the following subspaces of $\mathbb{R}^4$:
   (a) $W_1 = \{ x_1 + x_2 + x_3 + x_4 = 0 \}$.
   (b) $W_2 = \{ x_1 - x_2 = x_3 - x_4 = 0 \}$.

Solution. Choose a basis

\[ \{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\} = \{v_1, v_2, v_3\} \]

of $W_1$ and apply Gram-Schmidt to it:

\[
\begin{align*}
w_1 &= v_1 = (1, -1, 0, 0) \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = (1, 1, 2, -2, -1, 0) \\
w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 = (1, 1, 1, 1, 1, -1).
\end{align*}
\]

We obtain an orthogonal basis

\[ \{(1, -1, 0, 0), (\frac{1}{2}, \frac{1}{2}, -1, 0), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)\} \]

of $W_1$.

Choose a basis

\[ \{(1, 1, 0, 0), (0, 0, 1, 1)\} \]

of $W_2$, which is already orthogonal. \hfill \square

(2) Orthogonally diagonalize the following symmetric matrices $A$,
   i.e., find an orthogonal matrix $Q$ such that $Q^T AQ$ is diagonal.

a) \[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

Solution. a) The characteristic polynomial of $A$ is

\[
\det(xI - A) = \det \begin{bmatrix}
x - 1 & -2 \\
-2 & x - 1
\end{bmatrix} = (x + 1)(x - 3).
\]

\[\text{http://www.math.ualberta.ca/~xichen/math22514f/hw10sol.pdf}\]
So it has two eigenvalues $-1$ and $3$ with eigenspaces

\[
\text{Nul}(-I - A) = \text{Nul} \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}
\]

\[
\text{Nul}(3I - A) = \text{Nul} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.
\]

Normalizing the two eigenvectors, we obtain

\[
Q = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \quad \text{and} \quad Q^T AQ = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.
\]

b) The characteristic polynomial of $A$ is

\[
\det(xI - A) = \det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} = \det \begin{bmatrix} 0 & x^2 - 1 & -x - 1 \\ -1 & x & -1 \\ 0 & -x - 1 & x + 1 \end{bmatrix} = (x + 1)^2(x - 2).
\]

So it has eigenvalues $-1$ and $2$ with eigenspaces

\[
\text{Nul}(-I - A) = \text{Nul} \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

\[
\text{Nul}(2I - A) = \text{Nul} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.
\]

Applying Gram-Schmidt to $\text{Nul}(-I - A)$, we obtain

\[
\text{Nul}(-I - A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.
\]

After normalizing, we obtain

\[
Q = \begin{bmatrix} \sqrt{2} & \sqrt{6} & \sqrt{3} \\ -\sqrt{2} & \sqrt{6} & \sqrt{3} \\ 0 & -\sqrt{2} & \sqrt{3} \end{bmatrix} \quad \text{and} \quad Q^T AQ = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.
\]

(3) Which of the following statements are true and which are false? Justify your answer.
(a) Two symmetric matrices with the same characteristic polynomial must be similar.

Proof. True. Let $A$ and $B$ be two symmetric matrices with the same characteristic polynomial $f(x)$. Since every symmetric matrix is diagonalizable, $f(x)$ has $n$ real roots, counted with multiplicity. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the $n$ roots of $f(x)$. Then $A$ and $B$ are both similar to

$$
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}
$$

and hence they are similar. \qed

(b) The sum of two $n \times n$ orthogonal matrices is orthogonal.

Proof. False. Let $A = B = I$. Then $A$ and $B$ are orthogonal but $A + B = 2I$ is not. \qed

(c) The product of two $n \times n$ orthogonal matrices is orthogonal.

Proof. True. For $A$ and $B$ orthogonal,

$$(AB)^T(AB) = B^T(A^T A)B = B^T B = I$$

and hence $AB$ is orthogonal. \qed

(d) The sum of two $n \times n$ orthogonally diagonalizable matrices is also orthogonally diagonalizable.

Proof. True since a matrix is orthogonally diagonalizable if and only if it is symmetric and the sum of two symmetric matrices is symmetric. \qed

(4) Let $A$ be an $m \times n$ matrix satisfying $AA^T = I$. Show that $I - 2A^T A$ is a symmetric orthogonal matrix.

Proof. Since

$$(I - 2A^T A)^T = I^T - 2(A^T A)^T = I - 2A^T A,$$

$I - 2A^T A$ is symmetric.
Since
\[(I - 2A^T A)^T (I - 2A^T A) = (I - 2A^T A)^2\]
\[= I - 4A^T A + 4(A^T A)(A^T A)\]
\[= I - 4A^T A + 4A^T (AA^T)A\]
\[= I - 4A^T A + 4A^T A = I,\]

\(I - 2A^T A\) is orthogonal. In conclusion, \(I - 2A^T A\) is a symmetric orthogonal matrix. □

(5) Let \(T : \mathbb{R}^n \to \mathbb{R}^n\) be a linear transformation satisfying
\[\langle T(u), v \rangle = \langle u, T(v) \rangle\]
for all \(u, v \in \mathbb{R}^n\). Show that \([T]_{B \leftarrow B}\) is a symmetric matrix for every orthonormal basis \(B\) of \(\mathbb{R}^n\).

Proof. Let \(B = \{v_1, v_2, ..., v_n\}\) be an orthonormal basis of \(\mathbb{R}^n\). Then
\[v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + ... + \langle v, v_n \rangle v_n\]
for all \(v \in \mathbb{R}^n\). Therefore,
\[[v]_B = \begin{bmatrix} \langle v, v_1 \rangle \\ \langle v, v_2 \rangle \\ \vdots \\ \langle v, v_n \rangle \end{bmatrix}\]
for all \(v \in \mathbb{R}^n\). Consequently,
\[[T]_{B \leftarrow B} = \begin{bmatrix} [T(v_1)]_B \\ [T(v_2)]_B \\ \vdots \\ [T(v_n)]_B \end{bmatrix}\]
\[= \begin{bmatrix} \langle v_1, T(v_1) \rangle & \langle v_1, T(v_2) \rangle & \cdots & \langle v_1, T(v_n) \rangle \\ \langle v_2, T(v_1) \rangle & \langle v_2, T(v_2) \rangle & \cdots & \langle v_2, T(v_n) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, T(v_1) \rangle & \langle v_n, T(v_2) \rangle & \cdots & \langle v_n, T(v_n) \rangle \end{bmatrix}\]
\[= \begin{bmatrix} \langle v_i, T(v_j) \rangle \end{bmatrix}.
\]
That is, if we let \([T]_{B \leftarrow B} = [a_{ij}], a_{ij} = \langle v_i, T(v_j) \rangle\) for \(i, j = 1, 2, ..., n\).
Since \(\langle T(u), v \rangle = \langle u, T(v) \rangle\) for all \(u, v \in \mathbb{R}^n\),
\[a_{ij} = \langle v_i, T(v_j) \rangle = \langle T(v_i), v_j \rangle = \langle v_j, T(v_i) \rangle = a_{ji}\]
for all \(i, j\). Therefore, \([T]_{B \leftarrow B}\) is symmetric. □