Math 225 Final Review

Some information on the final:

- Time and location: 09:00-11:00 Thu Dec 11, ED GYM Rows 1,3,5,7,9,11 (Seats 1-10)
- Sections covered by the final: (Poole’s Book) 4.1, 4.3, 4.4, 5.1-5.4, 6.1-6.6

A list of topics covered by the final:

- Vector Space
- Subspace
- Null, Row and Column Spaces
- Linear Dependence
- Span, Basis and Dimension
- Linear Transformation
- Matrix Representation of Linear Transformation
- Kernel and Range of Linear Transformation
- Injectivity, Surjectivity and Bijectivity of Linear Transformation
- Rank and Rank Theorem
- Change of Basis
- Eigenvalue, Eigenvector and Characteristic Polynomial
- Diagonalization
- Inner Product, Norm and Orthogonality
- Orthogonal Complement and Orthogonal Projection
- Gram-Schmidt Process
- Orthogonal Diagonalization of Symmetric Matrices

Review problems:

1. Which of the following statements are true and which are false? Justify your answer.

   a) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three vectors in a vector space $V$. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, then $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

   b) Let $T: V \rightarrow W$ be a linear transformation between two vectors space $V$ and $W$ and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be $n$ vectors in $V$. If $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.
(c) Let \( T : V \to W \) be a linear transformation between two vectors space \( V \) and \( W \) and let \( v_1, v_2, \ldots, v_n \) be \( n \) vectors in \( V \). If \( v_1, v_2, \ldots, v_n \) are linearly dependent, then \( T(v_1), T(v_2), \ldots, T(v_n) \) are linearly dependent.

(d) An orthogonal matrix must be symmetric.

(e) Let \( T_1 : \mathbb{R}^5 \to \mathbb{R}^4 \) and \( T_2 : \mathbb{R}^4 \to \mathbb{R}^5 \) be two linear transformations. Then \( T_2 \circ T_1 \) cannot be onto.

(f) Let \( T_1 : \mathbb{R}^5 \to \mathbb{R}^4 \) and \( T_2 : \mathbb{R}^4 \to \mathbb{R}^5 \) be two linear transformations. Then \( T_1 \circ T_2 \) cannot be 1-1.

(g) Two \( 4 \times 4 \) matrices with the same characteristic polynomial must be similar.

(h) A symmetric matrix with characteristic polynomial \((x-1)^n\) must be the \( n \times n \) identity matrix.

(i) If \( V_1 \) and \( V_2 \) are two subspaces of \( \mathbb{R}^n \) satisfying \( V_1 \subset V_2^\perp \), then \( \dim V_1 + \dim V_2 \leq n \).

(j) For all linear transformations \( T : V \to V \), \( K(T^2) \subset K(T) \).

(k) Let \( T : V \to V \) be a linear transformation satisfying \( R(T^2) = R(T^3) \). Then \( R(T^n) = R(T^{n+1}) \) for all \( n \geq 2 \).

(l) The product of two orthogonally diagonalizable matrices must be orthogonally diagonalizable.

(m) \( R(T) \subset R(T \circ S) \) for all linear transformations \( S : U \to V \) and \( T : V \to W \).

(2) Let \( M_{m \times n}(\mathbb{R}) \) be the vector space of \( m \times n \) real matrices and \( T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \) be the map given by

\[
T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} A
\]

for all \( A \in M_{2 \times 2}(\mathbb{R}) \).

(a) Show that \( T \) is a linear transformation.

(b) Find the kernel, range and rank of \( T \).

(c) Is \( T \) onto, 1-1 and/or bijective? Justify your answer.

(d) Find the characteristic polynomial, eigenvalues and eigenvectors of \( T \).

(e) Is \( T \) diagonalizable? If it is, find a basis \( B \) of \( M_{2 \times 2}(\mathbb{R}) \) such that \([T]_{B \to B}\) is diagonal.

(3) Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation given by

\[
T(x, y, z) = (2x + y + z, x + 2y + z, x + y + 2z).
\]

(a) Find the kernel, range and rank of \( T \).

(b) Is \( T \) onto, 1-1 and/or bijective?
(c) Find the matrix $[T]_{B \rightarrow B}$ representing $T$ under the standard basis $B$.
(d) Find the matrix $[T]_{C \rightarrow C}$ representing $T$ under the basis

\[ C = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 0
\end{bmatrix} \].

(e) Find the eigenvalues, eigenvectors and characteristic polynomial of $T$.
(f) Is $T$ diagonalizable? If it is, find a basis $D$ such that $[T]_{D \rightarrow D}$ is diagonal.

(4) Let $P_3$ be the vector space of real polynomials of degree $\leq 3$ and $T : P_3 \rightarrow P_3$ be the map given by

\[ T(f(x)) = x f'(x) + f(1). \]

(a) Show that $T$ is a linear transformation.
(b) Find the kernel, range and rank of $T$.
(c) Find the characteristic polynomial, eigenvalues and eigenvectors of $T$.
(d) Is $T$ diagonalizable? If it is, find a basis $B$ of $P_3$ such that $[T]_{B \rightarrow B}$ is diagonal.

(5) Let $V_1$ and $V_2$ be two subspaces of $\mathbb{R}^n$ satisfying $V_1 \subset V_2 ^\perp$. Show that

\[ \text{proj}_{V_1 + V_2} \mathbf{v} = \text{proj}_{V_1} \mathbf{v} + \text{proj}_{V_2} \mathbf{v} \]

for all $\mathbf{v} \in \mathbb{R}^n$.

(6) Let $W$ be the subspace of $\mathbb{R}^4$ given by

$W = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 = x_2 + x_3 + x_4 = 0\}$.

(a) Find an orthonormal basis for $W$.
(b) Find the projection of $\mathbf{v} = (1, 1, 1, 1)$ onto $W$.
(c) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation given by

\[ T(\mathbf{v}) = \text{proj}_W \mathbf{v}. \]

Find the kernel and range of $T$ and find $[T]_{B \rightarrow B}$ under the standard basis $B$ of $\mathbb{R}^4$.

(7) Let $T : V \rightarrow V$ be a linear endomorphism whose characteristic polynomial is $x^4 - 3x^2 + 2$.

(a) Show that $T$ is bijective.
(b) Find the characteristic polynomial of $T + T^{-1}$.
(8) Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be two linear transformations between two vector spaces $V$ and $W$. Show that

$$K(T_1) \cap K(T_1 - T_2) = K(T_1 + T_2) \cap K(T_2).$$

(9) Find the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

which minimizes

$$\left\| \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right\|.$$

(10) Let $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be two linear transformations satisfying $T_1 \circ T_2 = T_2 \circ T_1 = 0,$

$$T_1(1, 1) = (2, 1) \text{ and } T_2(1, 2) = (1, 0).$$

(a) Find $T_1$ and $T_2$.

(b) Find the kernels and ranges of $T_1$ and $T_2$.

(11) Orthogonally diagonalize the following symmetric matrices:

a) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

(12) Let $V$ be a vector space of finite dimension and $T : V \rightarrow V$ be a linear transformation satisfying $T^2 - 3T + 2I = 0$. Show that $T$ is diagonalizable.