Outline

1. Orthogonal Matrix and Orthogonal Diagonalization
Orthogonal Matrix

Definition. A square matrix $A$ is orthogonal if $A^T = A^{-1}$, i.e.,
$A^T A = AA^T = I$.

- The identity matrix $I$ is orthogonal.
- $A$ is orthogonal if and only if $A^T$ is orthogonal.
- A product of orthogonal matrices is orthogonal:

$$(AB)^T (AB) = (B^T A^T) (AB) = B^T (A^T A) B = B^T B = I$$

for $A, B$ orthogonal.

- For an $n \times n$ orthogonal matrix $A$,

$$\langle Au, Av \rangle = (Av)^T Au = v^T (A^T A) u = v^T u = \langle u, v \rangle.$$

Or equivalently,

$$\langle T(u), T(v) \rangle = \langle u, v \rangle$$

for $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $T(v) = Av$. In particular,
$\| T(v) \| = \| v \|$.  

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Linear Algebra II Lecture 17
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Theorem

An $n \times n$ matrix $A$ is orthogonal if and only if the column (row) vectors form an orthonormal basis of $\mathbb{R}^n$.

Proof. Let $A = [v_1 \ v_2 \ \ldots \ v_n]$. Then

$$I = A^T A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} = [v_i^T v_j] = [\langle v_i, v_j \rangle]$$

$$\Rightarrow \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
Orthogonal Matrix and Orthogonal Diagonalization

Orthogonal Matrix and Orthonormal Basis

Theorem

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\[ \Rightarrow \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
Definition. We say two $n \times n$ matrices $A$ and $B$ are \textit{orthogonally similar} if there exists an orthogonal matrix $Q$ such that $B = Q^T AQ = Q^{-1} AQ$.

Obviously, if $A$ and $B$ are orthogonally similar, they are similar. But the converse is false. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}. $$

Since both $A$ and $B$ have two distinct eigenvalues $\pm 1$, they are similar. But they are not orthogonally similar. Otherwise, there exists an orthogonal matrix $Q$ such that $B = Q^T AQ$ and then

$$B = Q^T AQ \Rightarrow B^T = (Q^T AQ)^T = Q^T A^T Q = Q^T AQ = B $$

since $A$ is symmetric. But $B$ is not symmetric.
Orthogonal Similarity

Definition. We say two \( n \times n \) matrices \( A \) and \( B \) are **orthogonally similar** if there exists an orthogonal matrix \( Q \) such that \( B = Q^T AQ = Q^{-1} AQ \).

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Orthogonal Diagonalization

A square matrix $A$ is *orthogonally diagonalizable* if there exists an orthogonal matrix $Q$ such that $Q^T AQ$ is diagonal. Obviously, if $A$ is orthogonally diagonalizable, $Q^T AQ = D$ is diagonal and hence

$$A = QDQ^T \Rightarrow A^T = (QDQ^T)^T = QD^T Q^T = QDQ^T = A,$$

i.e., $A$ must be symmetric.

If $Q^T AQ$ is diagonal for $Q = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$ orthogonal, then

$$A \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

implies that $A v_k = \lambda_k v_k$ and hence $A$ has $n$ orthonormal eigenvectors.
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Orthogonal Matrix and Orthogonal Diagonalization

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Theorem

A square matrix $A$ is orthogonally diagonalizable if and only if $A$ is symmetric. That is, every $n \times n$ symmetric matrix $A$ has $n$ orthonormal eigenvectors $v_1, v_2, \ldots, v_n$ and

$$Q^T A Q = \begin{bmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_n
\end{bmatrix}$$

for $Q = [v_1 \ v_2 \ \ldots \ v_n]$

where $\lambda_k$ is the eigenvalue corresponding to $v_k$ for $k = 1, 2, \ldots, n$.

“Rough Algorithm”. Apply Gram-Schmidt to $n$ linearly independent eigenvectors $\{v_1, v_2, \ldots, v_n\}$ of $A$. 

Orthogonal Matrix and Orthogonal Diagonalization

Diagonalization of Symmetric Matrices

**Theorem**

A square matrix $A$ is orthogonally diagonalizable if and only if $A$ is symmetric. That is, every $n \times n$ symmetric matrix $A$ has $n$ orthonormal eigenvectors $v_1, v_2, ..., v_n$ and

$$Q^T AQ = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

for $Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

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“Rough Algorithm”. Apply Gram-Schmidt to $n$ linearly independent eigenvectors $\{v_1, v_2, ..., v_n\}$ of $A$. 
Let \( A \) be an \( n \times n \) symmetric matrix. Then

\[
x^T A y = \langle x, Ay \rangle = (x^T A y)^T = y^T A x = \langle Ax, y \rangle
\]

That is, the linear transformation \( T(x) = Ax \) is a self-adjoint operator satisfying \( \langle T(x), y \rangle = \langle x, T(y) \rangle \).

**Theorem**

Two eigenvectors of a symmetric \( A \) corresponding to different eigenvalues must be orthogonal.

**Proof.** Let \( Ax = ax \) and \( Ay = by \) for \( a \neq b \). Then

\[
\begin{align*}
\langle Ax, y \rangle &= \langle ax, y \rangle = a \langle x, y \rangle \\
\langle x, Ay \rangle &= \langle x, by \rangle = b \langle x, y \rangle
\end{align*}
\]

\[\Rightarrow a \langle x, y \rangle = b \langle x, y \rangle \Rightarrow \langle x, y \rangle = 0.\]
Let $A$ be an $n \times n$ symmetric matrix. Then

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\end{align*}$$

$$\Rightarrow a \langle x, y \rangle = b \langle x, y \rangle \Rightarrow \langle x, y \rangle = 0.$$
Algorithm of Orthogonal Diagonalization

For an $n \times n$ symmetric matrix $A$ with $m$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, we apply Gram-Schmidt to obtain an orthonormal basis $B_k$ for each eigenspace $\text{Nul}(\lambda_k I - A)$:

$$\mathbb{R}^n = \text{Nul}(\lambda_1 I - A) + \text{Nul}(\lambda_2 I - A) + \ldots + \text{Nul}(\lambda_m I - A)$$

$$= \text{Span } B_1 + \text{Span } B_2 + \ldots + \text{Span } B_m.$$  

Since

$$\text{Nul}(\lambda I - A) \subset \text{Nul}(\lambda' I - A)^\perp$$ for all $\lambda \neq \lambda'$,

$$B = B_1 \cup B_2 \cup \ldots \cup B_m = \{v_1, v_2, \ldots, v_n\}$$ is an orthonormal set of $n$ eigenvectors of $A$ and $Q^T A Q$ is diagonal for

$$Q = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}.$$
Example. Orthogonally diagonalize

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}.$$ 

Solution. $A$ has eigenvalues $-1$ and $7$ with eigenspaces

$$\text{Nul}(-I - A) = \text{Nul} \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{Nul}(7I - A) = \text{Nul} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalizing these two vectors, we obtain

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad Q^T A Q = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}.$$
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Normalizing these two vectors, we obtain

\[ Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad Q^T AQ = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}. \]
Example. Orthogonally diagonalize

\[ A = \begin{bmatrix}
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 1 \\
\end{bmatrix} \]

Solution. \( A \) has eigenvalues 5 and \(-1\) with eigenspaces

\[
\text{Nul}(5I - A) = \text{Span}\{(1, 1, 1)\}
\]

\[
\text{Nul}(-I - A) = \text{Span}\{(1, -1, 0), (0, 1, -1)\}
\]

Applying Gram-Schmidt to \( \text{Nul}(-I - A) \) and normalizing, we obtain

\[
Q = \begin{bmatrix}
\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{3} \\
\frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{3} \\
\frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3}
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0 & -1 \end{bmatrix} \).
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and

\[
Q^T A Q = \begin{bmatrix}
5 & -1 \\
-1 & -1
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\]
Proof that Symmetric Matrices are Orthogonally Diagonalizable

Theorem

*Every symmetric matrix $A$ has at least one eigenvector.*

proof. Consider the function $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.$$ 

Suppose that it achieves a minimum at $x = x_0$. Fixing $y \in \{x_0\}^\perp$, the function

$$g(t) = f(x_0 + ty) = \frac{\langle A(x_0 + ty), x_0 + ty \rangle}{\langle x_0 + ty, x_0 + ty \rangle}$$

has a minimum at $t = 0$. 

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Linear Algebra II Lecture 17
Orthogonal Matrix and Orthogonal Diagonalization

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Proof (CONT). Then

\[
g'(0) = 0 \Rightarrow \left( \frac{\langle A(x_0 + ty), x_0 + ty \rangle}{\langle x_0 + ty, x_0 + ty \rangle} \right)' \bigg| _{t=0} = 0
\]

\[
\Rightarrow \left( \frac{\langle Ax_0, x_0 \rangle + 2t\langle Ax_0, y \rangle + t^2\|y\|^2}{\|x_0\|^2 + t^2\|y\|^2} \right)' \bigg| _{t=0} = 0
\]

\[
\Rightarrow \langle Ax_0, y \rangle = 0
\]

for all \( y \in \{x_0\}^\perp \). Therefore, \( Ax_0 \in \text{Span}\{x_0\} \). That is,

\[
Ax_0 = \lambda x_0.
\]
Proof that Symmetric Matrices are Orthogonally Diagonalizable

**Theorem**

*Every symmetric matrix $A$ can be orthogonally diagonalized.*

Proof. Since $A$ has at least one eigenvector, $A v_1 = \lambda_1 v_1$ for some unit vector $v_1$. Using Gram-Schmidt, we can find an orthonormal basis \( \{v_1, v_2, \ldots, v_n\} \) of $\mathbb{R}^n$ and let

\[
P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}.
\]

Then

\[
P^T A P = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} A \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}
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*Every symmetric matrix $A$ can be orthogonally diagonalized.*

Proof. Since $A$ has at least one eigenvector, $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ for some unit vector $\mathbf{v}_1$. Using Gram-Schmidt, we can find an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ and let

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}.$$  

Then

$$P^TAP = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}.$$
Proof (CONT).

\[ P^T A P = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} A v_1 & A v_2 & \ldots & A v_n \end{bmatrix} = \begin{bmatrix} v_1^T A v_1 & * & \ldots & * \\ v_2^T A v_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & B & \vdots \\ v_n^T A v_1 & & \ldots & B \end{bmatrix} = \begin{bmatrix} \lambda_1 \ast & \ldots & \ast \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & B \\ 0 & & \ldots & B \end{bmatrix} \]

And since \((P^T A P)^T = P^T A^T P = P^T A P\), \(P^T A P\) is symmetric.
Proof (CONT). So

$$P^T A P = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & B \\ 0 & \ldots & 0 \end{bmatrix}$$

and $B$ is an $(n-1) \times (n-1)$ symmetric matrix. Continue this process on $B$ and we can prove the theorem by induction.

In conclusion, every $n \times n$ symmetric matrix is orthogonally diagonalizable; or equivalently, every $n \times n$ symmetric matrix $A$ has $n$ orthonormal eigenvectors.
Proof that Symmetric Matrices are Orthogonally Diagonalizable

Proof (CONT). So

\[ P^T A P = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & B \\ 0 & \ldots & 0 & 0 \end{bmatrix} \]

and \( B \) is an \((n - 1) \times (n - 1)\) symmetric matrix. Continue this process on \( B \) and we can prove the theorem by induction.

In conclusion, every \( n \times n \) symmetric matrix is orthogonally diagonalizable; or equivalently, every \( n \times n \) symmetric matrix \( A \) has \( n \) orthonormal eigenvectors.