Outline

1. Operations of Linear Transformations
2. Kernel and Range
**Definition**

Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations between two vector spaces $V$ and $W$ over $\mathbb{R}$. Then $T_1 + T_2 : V \to W$ is the map

$$(T_1 + T_2)(v) = T_1(v) + T_2(v).$$

Let $T : V \to W$ be a linear transformation between two vector spaces $V$ and $W$ over $\mathbb{R}$ and $c \in \mathbb{R}$. Then $cT : V \to W$ is the map

$$(cT)(v) = cT(v).$$

Let $T_1 : V \to W$ and $T_2 : U \to V$ be linear transformations between vector spaces $U$, $V$ and $W$. Then $T_1 \circ T_2$ is the map

$$(T_1 \circ T_2)(u) = T_1(T_2(u)).$$
Definition

Let \( T_1 : V \to W \) and \( T_2 : V \to W \) be linear transformations between two vector spaces \( V \) and \( W \) over \( \mathbb{R} \). Then \( T_1 + T_2 : V \to W \) is the map

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(T_1 + T_2)(v) = T_1(v) + T_2(v).
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Let \( T : V \to W \) be a linear transformation between two vector spaces \( V \) and \( W \) over \( \mathbb{R} \) and \( c \in \mathbb{R} \). Then \( cT : V \to W \) is the map

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(cT)(v) = cT(v).
\]

Let \( T_1 : V \to W \) and \( T_2 : U \to V \) be linear transformations between vector spaces \( U, V \) and \( W \). Then \( T_1 \circ T_2 \) is the map

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(T_1 \circ T_2)(u) = T_1(T_2(u)).
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Definition

Let $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ be linear transformations between two vector spaces $V$ and $W$ over $\mathbb{R}$. Then $T_1 + T_2 : V \rightarrow W$ is the map

$$(T_1 + T_2)(v) = T_1(v) + T_2(v).$$

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Let $T_1 : V \rightarrow W$ and $T_2 : U \rightarrow V$ be linear transformations between vector spaces $U$, $V$ and $W$. Then $T_1 \circ T_2$ is the map

$$(T_1 \circ T_2)(u) = T_1(T_2(u)).$$
Vector Space $L(V, W)$

**Theorem**

- For all linear transformations $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ and $c \in \mathbb{R}$, $T_1 + T_2$ and $cT_1$ are also linear transformations from $V$ to $W$.

- Furthermore,

  $$[T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1} \quad \text{and} \quad [cT_1]_{B_2 \leftarrow B_1} = c[T_1]_{B_2 \leftarrow B_1}$$

  where $B_1$ is a basis for $V$ and $B_2$ is a basis for $W$.

- Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$. Then $L(V, W)$ is itself a vector space over $\mathbb{R}$ under the addition and scalar multiplication defined above.
Vector Space \( L(V, W) \)

**Theorem**

- For all linear transformations \( T_1 : V \to W \) and \( T_2 : V \to W \) and \( c \in \mathbb{R} \), \( T_1 + T_2 \) and \( cT_1 \) are also linear transformations from \( V \) to \( W \).
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where \( B_1 \) is a basis for \( V \) and \( B_2 \) is a basis for \( W \).
- Let \( L(V, W) \) be the set of all linear transformations from \( V \) to \( W \). Then \( L(V, W) \) is itself a vector space over \( \mathbb{R} \) under the addition and scalar multiplication defined above.
Theorem

For all linear transformations $T_1 : V \rightarrow W$ and $T_2 : V \rightarrow W$ and $c \in \mathbb{R}$, $T_1 + T_2$ and $cT_1$ are also linear transformations from $V$ to $W$.

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$$[T_1 + T_2]_{B_2 \leftarrow B_1} = [T_1]_{B_2 \leftarrow B_1} + [T_2]_{B_2 \leftarrow B_1} \text{ and}$$

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where $B_1$ is a basis for $V$ and $B_2$ is a basis for $W$.

Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$. Then $L(V, W)$ is itself a vector space over $\mathbb{R}$ under the addition and scalar multiplication defined above.
Let $T_1 : V \to W$ and $T_2 : U \to V$ be linear transformations between vector spaces $U$, $V$ and $W$. Then $T_1 \circ T_2$ is a linear transformation from $U \to W$.

Furthermore,

$$[T_1 \circ T_2]_{B_3 \leftarrow B_1} = [T_1]_{B_3 \leftarrow B_2}[T_2]_{B_2 \leftarrow B_1}$$

where $B_1$, $B_2$, $B_3$ are bases for $U$, $V$, $W$, respectively.

1. $(T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3)$
2. $c(T_1 \circ T_2) = (cT_1) \circ T_2 = T_1 \circ (cT_2)$
3. $T_1 \circ (T_2 + T_3) = T_1 \circ T_2 + T_1 \circ T_3$
4. $(T_1 + T_2) \circ T_3 = T_1 \circ T_3 + T_2 \circ T_3$. 

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Theorem

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- $(T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3)$
- $c(T_1 \circ T_2) = (cT_1) \circ T_2 = T_1 \circ (cT_2)$
- $T_1 \circ (T_2 + T_3) = T_1 \circ T_2 + T_1 \circ T_3$
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Theorem

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4. $(T_1 + T_2) \circ T_3 = T_1 \circ T_3 + T_2 \circ T_3$. 
Example of Operations of Linear Transformations

Let $T_1(x, y) = (x + y, x - y)$ and $T_2(x, y) = (y, x)$ be two linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$. Then

$$(T_1 + T_2)(x, y) = T_1(x, y) + T_2(x, y) = (x + 2y, 2x - y)$$

$$(2T_1)(x, y) = 2T_1(x, y) = (2x + 2y, 2x - 2y)$$

$$(2T_2)(x, y) = 2T_2(x, y) = (2y, 2x)$$

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Note that $T_1 \circ T_2 \neq T_2 \circ T_1$ !!!
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Note that $T_1 \circ T_2 \neq T_2 \circ T_1$ !!
Let $B$ be the standard basis. Then

$$[T_1] = [T_1]_{B \leftarrow B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad [T_2] = [T_2]_{B \leftarrow B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(T_1 + T_2)(x, y) = (x + 2y, 2x - y)$$

$$[T_1 + T_2] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = [T_1] + [T_2]$$

$$(2T_1)(x, y) = (2x + 2y, 2x - 2y) \quad \quad (2T_2)(x, y) = (2y, 2x)$$

$$[2T_1] = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} = 2[T_1] \quad \text{and} \quad [2T_2] = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = 2[T_2]$$
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$$([T_1] + [T_2]) (x, y) = (x + 2y, 2x - y)$$

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Example of Operations of Linear Transformations

\[ T_1 \circ T_2(x, y) = (x + y, -x + y) \]

\[ [T_1 \circ T_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [T_1][T_2] \]

\[ T_2 \circ T_1(x, y) = (x - y, x + y) \]

\[ [T_2 \circ T_1] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [T_2][T_1] \]

Note that

\[ [T_1][T_2] \neq [T_2][T_1] \Leftrightarrow T_1 \circ T_2 \neq T_2 \circ T_1 \]
Example of Operations of Linear Transformations

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Example of Operations of Linear Transformations

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Definition

Let $T : V \to W$ be a linear transformation from $V$ to $W$. The \textit{kernel} of $T$ is $K(T) = \ker(T) = \{ x \in V : T(x) = 0 \} \subset V$. The \textit{range} of $T$ is the image of $T$, i.e., \[ R(T) = T(V) = \{ T(x) : x \in V \} \subset W. \]

Theorem

Let $T : V \to W$ be a linear transformation from $V$ to $W$. Then $K(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

Let $T(x, y) = (x, x)$ be a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$. Then $K(T) = \{ (x, y) : T(x, y) = (0, 0) \} = \{ (x, y) : x = 0 \}$ and $R(T) = \{ T(x, y) \} = \{ (x, x) \} = \{ (x, y) : x - y = 0 \}$. 
Kernel and Range

Definition

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The kernel of $T$ is $\ker(T) = \{x \in V : T(x) = 0\} \subset V$. The range of $T$ is the image of $T$, i.e., $R(T) = T(V) = \{T(x) : x \in V\} \subset W$.

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Theorem

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**Definition**

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. The *kernel* of $T$ is $K(T) = \ker(T) = \{ \mathbf{x} \in V : T(\mathbf{x}) = 0 \} \subset V$. The *range* of $T$ is the image of $T$, i.e.,

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**Theorem**

Let $T : V \rightarrow W$ be a linear transformation from $V$ to $W$. Then $K(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

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Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
Proof that \( K(T) \) and \( R(T) \) are subspaces

**\( K(T) \) is a subspace of \( V \).**

Since \( T(0) = 0 \), \( 0 \in K(T) \). For all \( \mathbf{v}_1, \mathbf{v}_2 \in K(T) \), \( T(\mathbf{v}_1) = T(\mathbf{v}_2) = 0 \) and hence

\[
T(\mathbf{v}_1 + c\mathbf{v}_2) = T(\mathbf{v}_1) + cT(\mathbf{v}_2) = 0
\]

for all \( c \in \mathbb{R} \). Therefore, \( \mathbf{v}_1 + c\mathbf{v}_2 \in K(T) \).

**\( R(T) \) is a subspace of \( W \).**

Since \( T(0) \in R(T) \), \( 0 \in R(T) \). For all \( \mathbf{w}_1, \mathbf{w}_2 \in R(T) \), there exist \( \mathbf{v}_1, \mathbf{v}_2 \in V \) such that \( \mathbf{w}_1 = T(\mathbf{v}_1) \) and \( \mathbf{w}_2 = T(\mathbf{v}_2) \). Thus,

\[
\mathbf{w}_1 + c\mathbf{w}_2 = T(\mathbf{v}_1) + cT(\mathbf{v}_2) = T(\mathbf{v}_1 + c\mathbf{v}_2) \in R(T).
\]
Proof that $K(T)$ and $R(T)$ are subspaces

**$K(T)$ is a subspace of $V$.**

Since $T(0) = 0$, $0 \in K(T)$. For all $v_1, v_2 \in K(T)$, $T(v_1) = T(v_2) = 0$ and hence

$$T(v_1 + cv_2) = T(v_1) + cT(v_2) = 0$$

for all $c \in \mathbb{R}$. Therefore, $v_1 + cv_2 \in K(T)$.

**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
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**$R(T)$ is a subspace of $W$.**

Since $T(0) \in R(T)$, $0 \in R(T)$. For all $w_1, w_2 \in R(T)$, there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Thus,

$$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in R(T).$$
If $R(T)$ is finite-dimensional, then the dimension of $R(T)$ is called the *rank* of $T$, denoted by

$$\text{rank}(T) = \dim R(T) = \dim T(V).$$

Given a basis $B = \{v_1, v_2, ..., v_n\}$ of $V$, then the range of a linear transformation $T : V \to W$ is

$$R(T) = T(V) = \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\}.$$ 

Note that

$$\text{rank}(T) = \dim R(T) = \dim \text{Span}\{T(v_1), T(v_2), ..., T(v_n)\} \leq n = \dim V.$$
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\]

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by

$$T(v) = Av$$

for an $m \times n$ matrix $A$. Then

$$K(T) = \{v : T(v) = 0\} = \{v : Av = 0\} = \text{Nul}(A).$$

$$R(T) = \text{Span}\{T(e_1), T(e_2), \ldots, T(e_n)\}$$

$$= \text{Span}\{Ae_1, Ae_2, \ldots, Ae_n\} = \text{Col}(A).$$

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