Linear Algebra II Lecture 2

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Outline

1. Vector Space
2. Subspace
Definition of Vector Space

Definition. A real (\(\mathbb{R}\)-) vector space (or a vector space over \(\mathbb{R}\)) is a set \(V\) equipped with two operations

- vector addition: \(u + v \in V\) for \(u, v \in V\) (\(+ : V \times V \to V\))
- scalar multiplication: \(cu \in V\) for \(c \in \mathbb{R}\) and \(u \in V\) (\(\cdot : \mathbb{R} \times V \to V\)).

More precisely, we call \((V, +, \cdot)\) a vector space over \(\mathbb{R}\) if

- vector addition + is commutative and associative:
  1. \(u + v = v + u\) for all \(u, v \in V\)
  2. \((u + v) + w = u + (v + w)\) all \(u, v, w \in V\)
Definition of Vector Space Cont.

- there is a zero vector \(0 \in V\) satisfying
  \[u + 0 = u\] and \(0u = 0\) for all \(u \in V\).
- scalar multiplication is associative:
  \[(ab)u = a(bu)\] for all \(a, b \in \mathbb{R}, u \in V\).
- vector addition and scalar multiplication are distributive:
  \[(a + b)u = au + bu\] and \(a(u + v) = au + av\)
  for all \(a, b \in \mathbb{R}, u \in V\).
- \(1u = u\) for all \(u \in V\).
Examples of Vector Spaces

- \( \mathbb{R}^n \)
- \( \mathbb{R}[x] = \{ f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R} \} \)
- the set \( F(\mathbb{R}) \) of the functions \( f : \mathbb{R} \to \mathbb{R} \)
- the set \( F(D) \) of the functions \( f : D \to \mathbb{R} \)
- the set \( M_{m \times n}(\mathbb{R}) \) of \( m \times n \) matrices with real entries
Examples of non vector spaces

- $\mathbb{R}^2$ with scalar multiplication $\ast$ defined by

  $$c \ast (x, y) = (cx, c^2 y)$$

  Then $(\mathbb{R}^2, +, \ast)$ is not a vector space because distributive law fails:

  $$(a + b) \ast (x, y) = ((a + b)x, (a + b)^2 y)$$

  $$a \ast (x, y) + b \ast (x, y) = (ax, a^2 y) + (bx, b^2 y)$$

  $$= ((a + b)x, (a^2 + b^2)y)$$

  $$\Rightarrow (a + b) \ast (x, y) \not\equiv a \ast (x, y) + b \ast (x, y).$$
Examples of non vector spaces Cont.

- \( \mathbb{R} \) with vector addition \( \oplus \) defined by

\[
x_1 \oplus x_2 = 2x_1 + 2x_2
\]

Then \((\mathbb{R}, \oplus, \cdot)\) is not a vector space because associative law fails:

\[
(x_1 \oplus x_2) \oplus x_3 = (2x_1 + 2x_2) \oplus x_3 = 4x_1 + 4x_2 + 2x_3
\]

\[
x_1 \oplus (x_2 \oplus x_3) = x_1 \oplus (2x_2 + 2x_3) = 2x_1 + 4x_2 + 4x_3
\]

\[
\Rightarrow (x_1 \oplus x_2) \oplus x_3 \not\equiv x_1 \oplus (x_2 \oplus x_3).
\]
Definition of Subspace

Definition A. A (linear/vector) subspace $W$ of a vector space $V$ (over $\mathbb{R}$) is a subset of $V$ which is itself a vector space (over $\mathbb{R}$) under the vector addition and scalar multiplication on $V$.

Definition B. A non-empty subset $W \subset V$ of a vector space $V$ over $\mathbb{R}$ is a subspace if

- $W$ is closed under vector addition: $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$;
- $W$ is closed under scalar multiplication: $cw \in W$ for all $w \in W$ and all $c \in \mathbb{R}$.

Definition C. A non-empty subset $W \subset V$ of a vector space $V$ over $\mathbb{R}$ is a subspace if

- $w_1 + cw_2 \in W$ for all $w_1, w_2 \in W$ and all $c \in \mathbb{R}$. 
Theorem (Definition B $\iff$ Definition C)

Let $V$ be a vector space over $\mathbb{R}$ and $W$ be a non-empty subset of $V$. Then

$$w_1 + w_2 \in W, cw_2 \in W \text{ for all } w_1, w_2 \in W \text{ and } c \in \mathbb{R}$$

if and only if $w_1 + cw_2 \in W \text{ for all } w_1, w_2 \in W \text{ and } c \in \mathbb{R}$.

Remark. To prove that Statement P $\iff$ Statement Q, one has to prove that P $\implies$ Q and Q $\implies$ P.
Proof of Definition B ⇔ Definition C

Proof of ⇒.

\[ w_2 \in W, \; c \in \mathbb{R} \Rightarrow cw_2 \in W \]
\[ w_1 \in W \]
\[ \Rightarrow w_1 + cw_2 \in W \]

Proof of ⇐.

\[ w_1 + cw_2 \in W \Rightarrow w_1 + w_2 \in W \] by setting \( c = 1 \)

It remains to prove that \( cw_2 \in W \) for all \( w_2 \in W \) and \( c \in \mathbb{R}. \)
\[ W \neq \emptyset \Rightarrow \text{there is } w \in W \Rightarrow w + (-1)w = 0 \in W \Rightarrow \]
\[ 0 + cw_2 = cw_2 \in W. \]
Examples of subspaces

- $N_A = \{ \mathbf{x} : A\mathbf{x} = 0 \} \subset \mathbb{R}^n$, where $A$ is an $m \times n$ matrix, e.g.,

\[
\{(x, y) : 3x + 4y = 0\} = \left\{(x, y) : \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}\right\}
\]

\[
\{(x, y, z) : x - y + z = 3x + 4y + 5z = 0\}
= \left\{(x, y, z) : \begin{bmatrix} 1 & -1 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}
\]

$N_A$ is called null space of $A$. It is a subspace of $\mathbb{R}^n$ since $0 \in N_A$ and

\[
A\mathbf{x}_1 = A\mathbf{x}_2 = 0 \Rightarrow A(\mathbf{x}_1 + c\mathbf{x}_2) = 0
\]
More examples of subspaces

- \( R_A = \{Ax : x \in \mathbb{R}^n\} \) is a subspace of \( \mathbb{R}^m \) since \( 0 \in R_A \) and
  \[
  Ax_1 + cAx_2 = A(x_1 + cx_2) \in R_A.
  \]

- \( \mathbb{R}_{\leq n}[x] = \{f(x) \in \mathbb{R}[x] : \deg f(x) \leq n\} \) is a subspace of \( \mathbb{R}[x] \) since \( 0 \in \mathbb{R}_{\leq n}[x] \) and
  \[
  \deg f \leq n, \deg g \leq n \Rightarrow \deg(f + cg) \leq n.
  \]

- \( W = \{f(x) \in \mathbb{R}[x] : f(x_1) = f(x_2) = ... = f(x_n) = 0\} \) is a subspace of \( \mathbb{R}[x] \) since \( 0 \in W \) and
  \[
  \begin{aligned}
  f(x_1) = f(x_2) = ... = f(x_n) &= 0 \\
  g(x_1) = g(x_2) = ... = g(x_n) &= 0
  \end{aligned}
  \]
  \[
  \Rightarrow (f + cg)(x_1) = (f + cg)(x_2) = ... = (f + cg)(x_n) = 0
  \]
More examples of subspaces

- \( C(\mathbb{R}) = \{ f \mid f : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous} \} \) is a subspace of \( F(\mathbb{R}) = \{ f \mid f : \mathbb{R} \rightarrow \mathbb{R} \} \) since \( 0 \in C(\mathbb{R}) \) and
  
  \[
  f(x) \text{ and } g(x) \text{ are continuous on } \mathbb{R} \\
  \Rightarrow f(x) + cg(x) \text{ is continuous on } \mathbb{R}.
  \]

- \( C^1(\mathbb{R}) = \{ f : \mathbb{R} \rightarrow \mathbb{R} \mid f'(x) \text{ exists and is continuous} \} \) is a subspace of both \( C(\mathbb{R}) \) and \( F(\mathbb{R}) \) since \( 0 \in C^1(\mathbb{R}) \) and
  
  \[
  f'(x) \text{ and } g'(x) \text{ exist and are continuous} \\
  \Rightarrow (f(x) + cg(x))' \text{ exists and is continuous}.
  \]
Even more examples of subspaces

- \( W = \{ A \in M_{n \times n}(\mathbb{R}) : A = A^T \} \) is a subspace of \( M_{n \times n} \) since \( 0 \in W \) and

\[
A = A^T, B = B^T \Rightarrow (A + cB)^T = A^T + cB^T = A + cB
\]

- \( W = \{ A \in M_{n \times n}(\mathbb{R}) : -A = A^T \} \) is a subspace of \( M_{n \times n} \) since \( 0 \in W \) and

\[
-A = A^T, -B = B^T \Rightarrow (A + cB)^T = A^T + cB^T = -(A + cB).
\]

- \( U = \{ A = [a_{ij}]_{m \times n} : a_{ij} = 0 \text{ for all } i > j \} \) is a subspace of \( M_{m \times n}(\mathbb{R}) \) since \( 0 \in U \) and

\[
A = [a_{ij}], B = [b_{ij}] \in U \Rightarrow a_{ij} = b_{ij} = 0 \text{ for all } i > j
\]
\[
\Rightarrow a_{ij} + cb_{ij} = 0 \text{ for all } i > j \Rightarrow A + cB \in U.
\]