(1) Which of the following statements are true and which are false? Justify your answer.

(a) The product of two orthogonal \( n \times n \) matrices is orthogonal.

**Solution.** True. Let \( A \) and \( B \) be two orthogonal matrices of the same size. Then \( A^T A = B^T B = I \). It follows that \( (AB)^T (AB) = B^T (A^T A) B = B^T B = I \). So \( AB \) is orthogonal.

(b) Two matrices with the same characteristic polynomials are similar.

**Solution.** False. Let 
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]

Then \( \det(\lambda I - A) = \det(\lambda I - B) = (\lambda - 1)^2 \). But \( A \) and \( B \) are not similar. Otherwise, \( B = P^{-1} A P \) for an invertible matrix \( P \). But \( P^{-1} A P = P^{-1} P = I \) and \( B \neq I \). Contradiction.

(c) For two \( n \times n \) invertible matrices \( A \) and \( B \), \( AB \) and \( BA \) are similar.

**Solution.** True since \( AB = B^{-1}(BA)B \).

(d) \( \text{rank}(A) = \text{rank}(A^2) \) for every square matrix \( A \).

**Solution.** False. Let
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

Then \( \text{rank}(A) = 1 \) but \( A^2 = 0 \) and \( \text{rank}(A^2) = 0 \).

(e) For a Householder matrix \( A \) and two vectors \( u \) and \( v \), if \( Au = v \), then \( Av = u \).

**Solution.** True. Since \( A \) is a Householder matrix, \( A \) is symmetric and orthogonal. Therefore, \( A^2 = A^T A = I \). Consequently,
\[
Au = v \Rightarrow A^2 u = Av \Rightarrow u = Av
\]

(f) If \( A \) is diagonalizable, so is \( A^T \).

**Solution.** True. Since \( A \) is diagonalizable, there is an invertible matrix \( P \) such that \( PAP^{-1} = D \) is diagonal. It follows that
\[
D^T = (PAP^{-1})^T = (P^{-1})^T A^T P^T = (P^T)^{-1} A^T P^T
\]
Since $D = D^T$ is diagonal, $A^T$ is diagonalizable.

(2) Show that two symmetric matrices are similar if and only if they have the same characteristic polynomials.

Proof. Let $A$ and $B$ be two symmetric matrices.

Suppose that $A$ and $B$ are similar. Then $A = PBP^{-1}$ for an invertible matrix $P$. Then
\[
\det(\lambda I - A) = \det(\lambda I - PBP^{-1}) = \det(P(\lambda I - B)P^{-1}) = \det(\lambda I - B) \det(P^{-1}) = \det(\lambda I - B)
\]

Therefore, $A$ and $B$ have the same characteristic polynomials.

On the other hand, suppose that $A$ and $B$ have the same characteristic polynomials. Let
\[
\det(\lambda I - A) = \det(\lambda I - B) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)
\]

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $B$.

Since $A$ and $B$ are symmetric, they are diagonalizable. There exist invertible matrices $P$ and $Q$ such that
\[
PAP^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad QBQ^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}
\]

Hence
\[
PAP^{-1} = QBQ^{-1} \Rightarrow A = P^{-1}QBQ^{-1}P = (P^{-1}Q)B(P^{-1}Q)^{-1}
\]

So $A$ and $B$ are similar. \qed

(3) Let $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (0, -1, -1)$ be two vectors in $\mathbb{R}^3$.

(a) Find the hyperplane $W = \{a_1x_1 + a_2x_2 + a_3x_3 = 0\}$ such that $\mathbf{u}$ is the reflection of $\mathbf{v}$ with respect to $W$.

Solution. We have $W = \mathbf{w}^\perp$ where $\mathbf{w} = \mathbf{u} - \mathbf{v} = (1, 2, 2)$.

So $W = \{x_1 + 2x_2 + 2x_3 = 0\}$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection with respect to the hyperplane obtained in part (a). Find the matrix $[T]_B$ representing $T$ with respect to the standard basis $B$. 

Solution. The corresponding Householder matrix is

\[
[T]_B = I - 2 \frac{w}{||w||} \left( \frac{w}{||w||} \right)^T = I - \frac{2}{9} \begin{bmatrix}
1 & 2 & 2 \\
2 & 4 & 4 \\
2 & 4 & 4 \\
\end{bmatrix}
\]

(4) Let \( Q(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2 + (x_1 - x_2)^2 \) be a quadratic form in the three variables \( x_1, x_2 \) and \( x_3 \).

(a) Find a \( 3 \times 3 \) symmetric matrix \( A \) such that

\[ Q(x_1, x_2, x_3) = x^T A x, \]

where \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \).

Solution. We have

\[ Q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_3 + 2x_2x_3 = x^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} x \]

(b) Find a \( 3 \times 3 \) orthogonal matrix \( P \) such that \( P^T A P \) is diagonal.

Solution. Note that \( (1, 1, 1) \) and \( (1, -1) \) are orthogonal. Therefore,

\[ P^T = \begin{bmatrix}
1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\
1/\sqrt{2} & -1/\sqrt{2} & 0 \\
a & b & c \\
\end{bmatrix} \]

Since \( P^T \) is orthogonal, \( a+b+c = a-b = 0 \) and \( a^2 + b^2 + c^2 = 1 \). Therefore,

\[ P^T = \begin{bmatrix}
1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\
1/\sqrt{2} & -1/\sqrt{2} & 0 \\
1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\
\end{bmatrix} \]

and hence

\[ P = \begin{bmatrix}
1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\
1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\
1/\sqrt{3} & 0 & -2/\sqrt{6} \\
\end{bmatrix} \]
(c) Let \( x = Py \). What is \( Q \) in terms of \( y_1, y_2 \) and \( y_3 \), where

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]

Solution. We have

\[
Q(x_1, x_2, x_3) = x^T Ax = y^T P^T A P y = 3y_1^2 + 2y_2^2
\]

(d) Find the maximum and minimum values of \( Q(x_1, x_2, x_3) \) subject to the constraint \( x_1^2 + x_2^2 + x_3^2 = 1 \) and where these extreme values are achieved.

Solution. We have

\[
0 \leq 3y_1^2 + 2y_2^2 \leq 3(y_1^2 + y_2^2 + y_3^2)
\]

Therefore, \( Q_{\min} = 0 \) when \( y_1 = y_2 = 0 \) and \( y_3 = \pm 1 \), i.e.,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\pm 1
\end{bmatrix} = \pm \begin{bmatrix}
1/\sqrt{6} \\
1/\sqrt{6} \\
-2/\sqrt{6}
\end{bmatrix}
\]

and \( Q_{\max} = 3 \) when \( y_1 = \pm 1 \) and \( y_2 = y_3 = 0 \), i.e.,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
\pm 1 \\
0 \\
0
\end{bmatrix} = \pm \begin{bmatrix}
1/\sqrt{3} \\
1/\sqrt{3} \\
1/\sqrt{3}
\end{bmatrix}
\]

(5) Let \( A = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \).

(a) Find an invertible matrix \( P \) such that \( PAP^{-1} \) is diagonal.

Solution. The characteristic polynomial of \( A \) is

\[
\det(\lambda I - A) = \lambda(\lambda - 2).
\]

For \( \lambda = 0 \), the corresponding eigenspace is

\[
\text{null}(-A) = \text{Span}\{\begin{bmatrix}
1 \\
1
\end{bmatrix}\}
\]

For \( \lambda = 1 \), the corresponding eigenspace is

\[
\text{null}(2I - A) = \text{Span}\{\begin{bmatrix}
1 \\
-1
\end{bmatrix}\}
\]

Let

\[
P = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}^{-1} = -\frac{1}{2} \begin{bmatrix}
-1 & -1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
1/2 & 1/2 \\
1/2 & -1/2
\end{bmatrix}
\]
Then
\[
PAP^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}
\]

(b) Find \(A^{2009}\).

**Solution.** Since
\[
PA^{2009}P^{-1} = (PAP^{-1})^{2009} = \begin{bmatrix} 0 & 0 \\ 0 & 2^{2009} \end{bmatrix}
\]
we have
\[
A^{2009} = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 2^{2009} \end{bmatrix} P
\]
\[
= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2^{2009} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}
\]
\[
= \begin{bmatrix} 2^{2008} & -2^{2008} \\ -2^{2008} & 2^{2008} \end{bmatrix}
\]

Alternatively, we may apply the Cayley-Hamilton theorem: \(A^2 = 2A\). It follows that \(A^n = 2^{n-1}A\). Therefore,
\[
A^{2009} = 2^{2008}A = \begin{bmatrix} 2^{2008} & -2^{2008} \\ -2^{2008} & 2^{2008} \end{bmatrix}
\]

(6) Let \(P_2\) be the vector space of polynomials in \(x\) of degree at most 2 and let \(T: P_2 \to P_2\) be the map given by
\[
T(f(x)) = f(1-x).
\]

(a) Show that \(T\) is a linear transformation.

**Proof.** For all \(f(x)\) and \(g(x)\) in \(P_2\),
\[
T(f(x) + g(x)) = (f + g)(1-x) = f(1-x) + g(1-x) = T(f(x)) + T(g(x))
\]
For all \(f(x) \in P_2\) and \(\lambda \in \mathbb{R}\),
\[
T(\lambda f(x)) = \lambda f(1-x) = \lambda T(f(x))
\]
Therefore, \(T\) is a linear transformation. \(\square\)
(b) Find the matrix $[T]_S$ representing $T$ with respect to the basis $S = \{1, x, x^2\}$.

**Solution.** We have $T(1) = 1$, $T(x) = 1 - x$ and $T(x^2) = (1 - x)^2$. Therefore,

$$[T(1)]_S = [1]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x)] = [1 - x]_S = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and

$$[T(x^2)]_S = [(1 - x)^2]_S = [1 - 2x + x^2]_S = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

It follows that

$$[T]_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Let $B = \{1, x - 1, (x - 1)^2\}$. Find the transition matrix $P_{S \rightarrow B}$ and the matrix $[T]_B$ representing $T$ with respect to $B$.

**Solution.** Since

$$[1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [x]_B = [1 + (x - 1)]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$[x^2]_B = [(1 + (x - 1))^2]_B = [1 + 2(x - 1) + (x - 1)^2]_B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$[P_{S \rightarrow B}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Since

$$[T(1)]_B = [1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [T(x - 1)] = [-x]_B = [-1 - (x - 1)]_B = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$
and

\[ [T((x - 1)^2)]_B = [x^2]_B = [1 + 2(x - 1) + (x - 1)^2]_B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \]

we have

\[ [T]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \]

(d) Does there exist a basis \( B \) such that \( [T]_B \) is diagonal? If yes, find such a basis \( B \). If no, justify your answer.

**Solution.** Such a basis exists if and only if \( [T]_S \) is diagonalizable. The matrix \( [T]_S \) has two eigenvalues 1 and \(-1\). For \( \lambda = 1 \), the corresponding eigenspace is

\[ \text{null}(I - [T]_S) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \]

For \( \lambda = -1 \), the corresponding eigenspace is

\[ \text{null}(-I - [T]_S) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\} \]

Therefore, \( [T]_S \) has three linearly independent eigenvectors and is hence diagonalizable. The corresponding basis \( B \) is \( \{1, x - x^2, 1 - 2x\} \).

(7) Do the following:

(a) Find the least squares solution of

\[ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \]

**Solution.** The associated normal system is

\[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \]

So the least squares solution is

\[ \begin{cases} x_1 = 4 \\ x_2 = -4 \end{cases} \]
(b) Express \( w = (8, 0, 0) \) in the form \( w = w_1 + w_2 \), where \( w_1 \) lies in the subspace \( W \) of \( \mathbb{R}^3 \) spanned by the vectors \( u_1 = (1, 1, 1) \) and \( u_2 = (0, 1, 0) \) and \( w_2 \) is orthogonal to \( W \).

**Solution.** Note that

\[
    w_1 = \text{Proj}_W w = \text{Proj}_{\text{col}(A)} w
\]

where

\[
    A = \begin{bmatrix}
        1 & 0 \\
        1 & 1 \\
        1 & 0 
    \end{bmatrix}
\]

By part (a),

\[
    w_1 = x_1 u_1 + x_2 u_2 = (4, 0, 4)
\]

and

\[
    w_2 = w - w_1 = (4, 0, -4)
\]