Solutions for Math 217 Assignment #9

(1) Show that if the limit

\[ L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \]

exists, \( \lim_{n \to \infty} n \sqrt[1/n]{|a_n|} \) exists and

\[ \lim_{n \to \infty} n \sqrt[1/n]{|a_n|} = L. \]

Use this to conclude that the radius \( R \) of convergence of the power series \( \sum a_n x^n \) is \( \frac{1}{L} \) if \( L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \) exists. Here we follow the convention that \( R = \infty \) if \( L = 0 \).

Proof. Let \( b_n = \ln \frac{|a_n|}{|a_{n-1}|} = \ln |a_n| - \ln |a_{n-1}| \) for \( n \in \mathbb{Z}^+ \), where we set \( a_0 = 1 \) for convenience. Then

\[
\ln( n \sqrt[1/n]{|a_n|} ) = \frac{1}{n} \ln |a_n| \\
= \frac{1}{n} \sum_{k=1}^{n} (\ln |a_k| - \ln |a_{k-1}|) \\
= \frac{1}{n} \sum_{k=1}^{n} b_k
\]

Since \( \lim_{n \to \infty} b_n = \ln L \),

\[ \lim_{n \to \infty} \ln( n \sqrt[1/n]{|a_n|} ) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} b_k = \ln L \]

by Assignment #6 (5). So \( \lim_{n \to \infty} n \sqrt[1/n]{|a_n|} = L \). Consequently, the radius of convergence of \( \sum a_n x^n \) is

\[ R = \frac{1}{\limsup_{n \to \infty} n \sqrt[1/n]{|a_n|}} = \frac{1}{\lim_{n \to \infty} n \sqrt[1/n]{|a_n|}} = \frac{1}{L}. \]

□

(2) Let \( f(x) \) and \( g(x) \) be two continuous functions on \( [a,b] \) with continuous first derivatives \( f'(x) \) and \( g'(x) \) on \( (a,b) \). If \( g'(x) \neq 0 \) for all \( x \in (a,b) \), then there exists a number \( c \in (a,b) \) such that

\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \]
Proof 1. Let \( h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a)) \). Then \( h(x) \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( h(a) = h(b) \). Therefore, there exists \( c \in (a, b) \) such that \( h'(c) = 0 \) by Rolle’s Theorem. It follows that
\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.
\]

Proof 2. By the inverse function theorem, \( g^{-1} : [A, B] \to [a, b] \) exists as a continuous function on \([A, B]\) with continuous first derivative on \((A, B)\) given by
\[
(g^{-1})'(y) = \frac{1}{g'(g^{-1}(y))}
\]
for all \( y \in (A, B) \), where
\[
A = \min_{x \in [a, b]} g(x) \quad \text{and} \quad B = \max_{x \in [a, b]} g(x).
\]
Let \( h = f \circ g^{-1} \). Then \( h(x) \) is a continuous function on \([A, B]\) with continuous first derivative on \((A, B)\). By Mean Value Theorem, there exists a number \( C \in (A, B) \) such that
\[
\frac{h(B) - h(A)}{B - A} = h'(C).
\]
Since \( g(x) \) is monotonous, we have either \( A = g(a) \) and \( B = g(b) \) or \( A = g(b) \) and \( B = g(a) \). Therefore, we always have
\[
\frac{h(B) - h(A)}{B - A} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]
By chain rule,
\[
h'(y) = f'(g^{-1}(y))(g^{-1})'(y) = \frac{f'(g^{-1}(y))}{g'(g^{-1}(y))}
\]
and hence
\[
h'(C) = \frac{f'(g^{-1}(C))}{g'(g^{-1}(C))}.
\]
Let \( c = g^{-1}(C) \). Then
\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.
\]
\[\square\]
(3) Let $S$ be a convex set in $\mathbb{R}$. We call a function $f : S \to \mathbb{R}$ convex if
\[
f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)
\]
for all $t \in [0,1]$ and $x_1, x_2 \in S$.

(a) Show that $f : S \to \mathbb{R}$ is convex if and only if the set
\[
\{(x, y) : x \in S, y \geq f(x)\}
\]
is convex.

(b) Show that if $f : S \to \mathbb{R}$ is convex, then
\[
f(t_1x_1 + t_2x_2 + \ldots + t_nx_n) \leq t_1f(x_1) + t_2f(x_2) + \ldots + t_nf(x_n)
\]
for all $n \in \mathbb{Z}^+$, $x_1, x_2, \ldots, x_n \in S$ and $t_1, t_2, \ldots, t_n \geq 0$ with $\sum_{k=1}^n t_k = 1$.

Proof. (a) Suppose that $f$ is convex. Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be two points of the set $T = \{(x, y) : x \in S, y \geq f(x)\}$. Then $y_1 \geq f(x_1)$ and $y_2 \geq f(x_2)$. It follows that
\[
ty_1 + (1-t)y_2 \geq tf(x_1) + (1-t)f(x_2)
\]
for all $t \in [0,1]$. And since $f$ is convex,
\[
ty_1 + (1-t)y_2 \geq tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2)
\]
and hence $tp_1 + (1-t)p_2 = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in T$ for all $t \in [0,1]$. So $T$ is convex.

Suppose that $T$ is convex. Then $p_1 = (x_1, f(x_1)) \in T$ and $p_2 = (x_2, f(x_2)) \in T$ for all $x_1, x_2 \in S$. Since $T$ is convex,
\[
tp_1 + (1-t)p_2 = (tx_1 + (1-t)x_2, tf(x_1) + (1-t)f(x_2)) \in T
\]
and hence
\[
tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2)
\]
for all $t \in [0,1]$. Therefore, $f$ is convex.

(b) We prove this by induction on $n$. This is trivial for $n = 1, 2$. Suppose that this holds for $n$. We want to show that it also holds for $n + 1$, i.e.,
\[
f(t_1x_1 + t_2x_2 + \ldots + t_nx_n + t_{n+1}x_{n+1}) \leq t_1f(x_1) + t_2f(x_2) + \ldots + t_nf(x_n) + t_{n+1}f(x_{n+1})
\]
for all $x_1, x_2, \ldots, x_n, x_{n+1} \in S$ and $t_1, t_2, \ldots, t_n, t_{n+1} \geq 0$ with $\sum_{k=1}^{n+1} t_k = 1$. If $t_{n+1} = 1$, $t_1 = t_2 = \ldots = t_n = 0$ and the inequality is obvious. Suppose that $t_{n+1} < 1$. Let
\[
w = \frac{t_1}{1-t_{n+1}}x_1 + \frac{t_2}{1-t_{n+1}}x_2 + \ldots + \frac{t_n}{1-t_{n+1}}x_n.
\]
Since
\[
\frac{t_1}{1 - t_{n+1}} + \frac{t_2}{1 - t_{n+1}} + \ldots + \frac{t_n}{1 - t_{n+1}} = 1
\]
w \in S. And since f is convex,
\[
f((1 - t_{n+1})w + t_{n+1}x_{n+1}) \leq (1 - t_{n+1})f(w) + t_{n+1}f(x_{n+1})
\]
By induction hypothesis,
\[
f(w) = f \left( \frac{t_1}{1 - t_{n+1}} x_1 + \frac{t_2}{1 - t_{n+1}} x_2 + \ldots + \frac{t_n}{1 - t_{n+1}} x_n \right)
\leq \frac{t_1}{1 - t_{n+1}} f(x_1) + \frac{t_2}{1 - t_{n+1}} f(x_2) + \ldots + \frac{t_n}{1 - t_{n+1}} f(x_n).
\]
Therefore,
\[
f(t_1x_1 + t_2x_2 + \ldots + t_nx_n + t_{n+1}x_{n+1})
= f((1 - t_{n+1})w + t_{n+1}x_{n+1}) \leq (1 - t_{n+1})f(w) + t_{n+1}f(x_{n+1})
\leq (1 - t_{n+1}) \left( \frac{t_1}{1 - t_{n+1}} f(x_1) + \frac{t_2}{1 - t_{n+1}} f(x_2) + \ldots + \frac{t_n}{1 - t_{n+1}} f(x_n) \right)
+ t_{n+1}f(x_{n+1})
= t_1f(x_1) + t_2f(x_2) + \ldots + t_nf(x_n) + t_{n+1}f(x_{n+1}).
\]

(4) Let \(S\) be a convex set in \(\mathbb{R}\) and \(f : S \to \mathbb{R}\) be a convex function with continuous second derivative \(f''(x)\) on \(S\).

(a) Show that \(f\) is convex if and only if \(f''(x) \geq 0\) for all \(x \in S\).

(b) Show that \(f(x) = x^a\) is convex on \((0, \infty)\) for \(a > 1\) or \(a < 0\) and conclude that

\[
(t_1x_1^b + t_2x_2^b + \ldots + t_nx_n^b)^{1/b} \geq (t_1x_1^c + t_2x_2^c + \ldots + t_nx_n^c)^{1/c}
\]

for all \(n \in \mathbb{Z}^+, x_1, x_2, \ldots, x_n > 0, b > c > 0\) and \(t_1, t_2, \ldots, t_n \geq 0\) with \(\sum_{k=1}^n t_k = 1\).

Proof. (a) Suppose that \(f''(x) \geq 0\) for all \(x \in S\). Let \(x_1, x_2 \in S, x_1 < x_2\) and \(x_3 = tx_1 + (1-t)x_2\) for some \(t \in [0,1]\). By Mean Value Theorem, there exists \(c_1 \in [x_1, x_3]\) and \(c_2 \in [x_3, x_2]\) such that

\[
f'(c_1) = \frac{f(x_3) - f(x_1)}{x_3 - x_1} \quad \text{and} \quad f'(c_2) = \frac{f(x_2) - f(x_3)}{x_2 - x_3}.
\]
And since \( f''(x) \geq 0 \) for all \( x \in S \), \( f'(x) \) is nondecreasing and hence \( f'(c_1) \leq f'(c_2) \). Therefore,

\[
\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3}
\]

and it follows that

\[
tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2).
\]

Hence \( f \) is convex.

Suppose that \( f \) is convex and \( f''(x_0) < 0 \) for some \( x_0 \in S \). Since \( f''(x) \) is continuous, \( \{x : f''(x) < 0\} \) is open in \( S \). So there exists \( r > 0 \) such that \( f''(x) < 0 \) for all \( x \in I = B_r(x_0) \cap S \).

Let \( x_1, x_2 \in I \) and \( x_1 < x_2 \). Since \( B_r(x_0) \) and \( S \) are convex, \( I \) is convex and hence \((x_1, x_2) \subseteq I \) and \( f''(x) < 0 \) for all \( x \in (x_1, x_2) \). Let \( x_3 = (x_1 + x_2)/2 \). Mean Value Theorem, there exists \( c_1 \in (x_1, x_3) \) and \( c_2 \in (x_3, x_2) \) such that

\[
f'(c_1) = \frac{f(x_3) - f(x_1)}{x_3 - x_1} \quad \text{and} \quad f'(c_2) = \frac{f(x_2) - f(x_3)}{x_2 - x_3}.
\]

And since \( f''(x) < 0 \) for all \( x \in (x_1, x_2) \), \( f'(x) \) is decreasing in \((x_1, x_2)\) and hence \( f'(c_1) > f'(c_2) \). Therefore,

\[
\frac{f(x_3) - f(x_1)}{x_3 - x_1} > \frac{f(x_2) - f(x_3)}{x_2 - x_3}
\]

and it follows that

\[
\frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) < f \left( \frac{1}{2}x_1 + \frac{1}{2}x_2 \right).
\]

This contradicts the fact that \( f \) is convex.

(b) Since \( f''(x) = a(a-1)x^{a-2} > 0 \) for \( x > 0 \) and \( a \in (-\infty, 0) \cup (1, \infty) \), \( x^a \) is convex on \((0, \infty)\) for \( a > 1 \) or \( a < 0 \).

When \( b > c > 0 \), \( b/c > 1 \) and hence \( f(x) = x^{b/c} \) is convex. By (3) (b),

\[
(t_1y_1 + t_2y_2 + \ldots + t_ny_n)^{b/c} \leq t_1y_1^{b/c} + t_2y_2^{b/c} + \ldots + t_ny_n^{b/c}
\]

for all \( y_1, y_2, \ldots, y_n > 0 \) and \( t_1, t_2, \ldots, t_n \geq 0 \) with \( \sum_{k=1}^n t_k = 1 \).

Let \( x_k = y_k^{1/c} \) for \( k = 1, 2, \ldots, n \). Then

\[
(t_1x_1^{c} + t_2x_2^{c} + \ldots + t_nx_n^{c})^{b/c} \leq t_1x_1^{b} + t_2x_2^{b} + \ldots + t_nx_n^{b}
\]

and hence

\[
(t_1x_1^{c} + t_2x_2^{c} + \ldots + t_nx_n^{c})^{1/c} \leq (t_1x_1^{b} + t_2x_2^{b} + \ldots + t_nx_n^{b})^{1/b}
\]

for all \( x_1, x_2, \ldots, x_n > 0 \) and \( t_1, t_2, \ldots, t_n \geq 0 \) with \( \sum_{k=1}^n t_k = 1 \). \( \square \)
(5) Let
\[ f(x) = \left| x - 2 \left\lfloor \frac{x + 1}{2} \right\rfloor \right|. \]

(a) Show that \( f(x) \) is continuous on \( \mathbb{R} \) and differentiable everywhere on \( \mathbb{R} \setminus \mathbb{Z} \).

(b) Let \( c \) be a nonzero constant. Find all the points on \( \mathbb{R} \) where \( f(cx) \) is differentiable.

(c) Let
\[ F(x) = \sum_{n=1}^{\infty} \frac{f(nx)}{2^n}. \]

Show that \( F(x) \) is continuous on \( \mathbb{R} \) and differentiable everywhere on \( \mathbb{R} \setminus \mathbb{Q} \).

\textbf{Proof.} Notice that
\[
 f(x) = \begin{cases} 
 2n - x & \text{if } 2n - 1 \leq x < 2n \\
 x - 2n & \text{if } 2n \leq x < 2n + 1 
\end{cases}
\]
for \( n \in \mathbb{Z} \). For \( x_0 \notin \mathbb{Z} \), \( f(x) = 2n - x \) or \( f(x) = x - 2n \) in \( B_r(x_0) \) for some \( r > 0 \). So \( f(x) \) is continuous and differentiable everywhere on \( \mathbb{R} \setminus \mathbb{Z} \). Since
\[
 \lim_{x \to (2n)^-} f(x) = \lim_{x \to (2n)^+} f(x) = 0 = f(2n)
\]
and
\[
 \lim_{x \to (2n-1)^-} f(x) = \lim_{x \to (2n-1)^+} f(x) = 1 = f(2n - 1)
\]
for all \( n \in \mathbb{Z} \), \( f(x) \) is continuous at \( x \in \mathbb{Z} \). In summary, \( f(x) \) is continuous on \( \mathbb{R} \) and differentiable on \( \mathbb{R} \setminus \mathbb{Z} \).

Since
\[
 \lim_{h \to 0^+} \frac{f(2n + h) - f(2n)}{h} = 1 \neq -1 = \lim_{h \to 0^-} \frac{f(2n + h) - f(2n)}{h}
\]
and
\[
 \lim_{h \to 0^+} \frac{f(2n - 1 + h) - f(2n - 1)}{h} = -1
\]
\[
 \neq 1 = \lim_{h \to 0^-} \frac{f(2n - 1 + h) - f(2n - 1)}{h}
\]
for all \( n \in \mathbb{Z} \), \( f(x) \) is not differentiable at \( x \in \mathbb{Z} \). Therefore, \( f(cx) \) is differentiable on \( \mathbb{R} \setminus \{n/c : n \in \mathbb{Z}\} \).
Since $|f(x)| \leq 1$ for all $x \in \mathbb{R}$,
$$|f(nx)| \leq \frac{1}{2^n}$$
and $\sum 2^{-n}$ converges, the series $\sum f(nx)/2^n$ converges uniformly. And since $f(nx)$ is continuous on $\mathbb{R}$ for all $n$, $F(x)$ is continuous on $\mathbb{R}$.

To show that $F(x)$ is differentiable on $\mathbb{R}\setminus\mathbb{Q}$, we first show that
$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|$$
for all $x_1, x_2 \in \mathbb{R}$. WLOG, we may assume that $x_1 \leq x_2$. Since $|f(x_1) - f(x_2)| \leq 1$, this is obvious when $|x_1 - x_2| \geq 1$. So we assume that $x_1 \leq x_2 < x_1 + 1$.

When $2n \leq x_1 < 2n + 1$ for some integer $n \in \mathbb{Z}$, we have either $2n \leq x_2 < 2n + 1$ or $2n + 1 \leq x_2 < 2n + 2$. In either case, it is trivial to check that the inequality holds. When $2n - 1 \leq x_1 < 2n$ for some integer $n \in \mathbb{Z}$, we have either $2n - 1 \leq x_2 < 2n$ or $2n \leq x_2 < 2n + 1$. In either case, it is again trivial to check that the inequality holds. Therefore,
$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|$$
for all $x_1, x_2 \in \mathbb{R}$.

Let
$$G(x, h) = \frac{F(x + h) - F(x)}{h} = \sum_{n=1}^{\infty} \frac{f(n(x + h)) - f(nx)}{2^n h}$$
By the inequality above,
$$\left| \frac{f(n(x + h)) - f(nx)}{2^n h} \right| \leq \frac{n}{2^n}$$
for all $x, h \in \mathbb{R}$. And since the series $\sum n/2^n$ converges, $G(x, h)$ converges uniformly for all $x \in \mathbb{R}$ and $h \neq 0$. Therefore,
$$F'(x) = \lim_{h \to 0} G(x, h) = \sum_{n=1}^{\infty} \lim_{h \to 0} \frac{f(n(x + h)) - f(nx)}{2^n h} = \sum_{n=1}^{\infty} \frac{n f'(nx)}{2^n}$$
for all $x$ where $f'(nx)$ exists. And since $f(nx)$ is differentiable on $\mathbb{R}\setminus\{m/n : m \in \mathbb{Z}\}$, $F(x)$ is differentiable on $\mathbb{R}\setminus\mathbb{Q}$. □