Math 217 Assignment #9
Due Nov. 22, 2010

(1) Show that if the limit
\[ L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \]
exists, \( \lim_{n \to \infty} \sqrt[n]{|a_n|} \) exists and
\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = L. \]
Use this to conclude that the radius \( R \) of convergence of the power series \( \sum a_n x^n \) is \( \frac{1}{L} \) if \( L = \lim_{n \to \infty} |a_{n+1}|/|a_n| \) exists. Here we follow the convention that \( R = \infty \) if \( L = 0 \).

(2) Let \( f(x) \) and \( g(x) \) be two continuous functions on \([a, b]\) with continuous first derivatives \( f'(x) \) and \( g'(x) \) on \((a, b)\). If \( g'(x) \neq 0 \) for all \( x \in (a, b) \), then there exists a number \( c \in (a, b) \) such that
\[ f(b) - f(a) = \frac{f'(c)}{g'(c)} g(b) - g(a). \]

(3) Let \( S \) be a convex set in \( \mathbb{R} \). We call a function \( f : S \to \mathbb{R} \) convex if
\[ f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \]
for all \( t \in [0, 1] \) and \( x_1, x_2 \in S \).
(a) Show that \( f : S \to \mathbb{R} \) is convex if and only if the set \( \{ (x, y) : x \in S, y \geq f(x) \} \) is convex.
(b) Show that if \( f : S \to \mathbb{R} \) is convex, then
\[ f(t_1 x_1 + t_2 x_2 + \ldots + t_n x_n) \leq t_1 f(x_1) + t_2 f(x_2) + \ldots + t_n f(x_n) \]
for all \( n \in \mathbb{Z}^+, x_1, x_2, \ldots, x_n \in S \) and \( t_1, t_2, \ldots, t_n \geq 0 \) with \( \sum_{k=1}^n t_k = 1 \).

(4) Let \( S \) be a convex set in \( \mathbb{R} \) and \( f : S \to \mathbb{R} \) be a convex function with continuous second derivative \( f''(x) \) on \( S \).
(a) Show that \( f \) is convex if and only if \( f''(x) \geq 0 \) for all \( x \in S \).
(b) Show that \( f(x) = x^a \) is convex on \((0, \infty)\) for \( a > 1 \) or \( a < 0 \) and conclude that
\[ (t_1 x_1^b + t_2 x_2^b + \ldots + t_n x_n^b)^{1/b} \geq (t_1 x_1^c + t_2 x_2^c + \ldots + t_n x_n^c)^{1/c} \]
for all \( n \in \mathbb{Z}^+, x_1, x_2, \ldots, x_n > 0, b > c > 0 \) and \( t_1, t_2, \ldots, t_n \geq 0 \) with \( \sum_{k=1}^n t_k = 1 \).
(5) Let
\[ f(x) = \left| x - 2 \left\lfloor \frac{x + 1}{2} \right\rfloor \right|. \]

(a) Show that \( f(x) \) is continuous on \( \mathbb{R} \) and differentiable everywhere on \( \mathbb{R} \setminus \mathbb{Z} \).

(b) Let \( c \) be a nonzero constant. Find all the points on \( \mathbb{R} \) where \( f(cx) \) is differentiable.

(c) Let
\[ F(x) = \sum_{n=1}^{\infty} \frac{f(nx)}{2^n}. \]
Show that \( F(x) \) is continuous on \( \mathbb{R} \) and differentiable everywhere on \( \mathbb{R} \setminus \mathbb{Q} \).