(1) Let $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{Z}^+$. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^b$ converges for all $b \geq 1$.

(2) Find the radius of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} 2^n x^n$;

(b) $\sum_{n=0}^{\infty} n^2 x^n$.

(3) Let $S$ be a set in $\mathbb{R}^n$. Show that

(a) $\{(x, y) : x + y \in S\}$ is an open set in $\mathbb{R}^{2n}$ if $S$ is open in $\mathbb{R}^n$;

(b) $\{(x, y) : x - y \in S\}$ is a closed set in $\mathbb{R}^{2n}$ if $S$ is closed in $\mathbb{R}^n$.

(4) We call a function $f : S \to \mathbb{R}^n$ locally constant if for every point $p \in S$, there is $r > 0$ such that $f$ is constant in $S \cap B_r(p)$, i.e., $f(x_1) = f(x_2)$ for all $x_1, x_2 \in S \cap B_r(p)$, where $S$ is a set in $\mathbb{R}^m$. Show that locally constant functions are continuous.

(5) Show that a set $S \subset \mathbb{R}^n$ is connected if and only if every locally constant function $f : S \to \mathbb{R}$ is constant.
Math 217 Assignment #8  
Due Nov. 15, 2010

(1) Let \( f : S \to \mathbb{R}^n \) and \( g : S \to \mathbb{R}^n \) be two continuous functions defined on a set \( S \subset \mathbb{R}^m \). If there is a dense subset \( D \subset S \) (i.e. \( \overline{D} \supset S \)) such that \( f(x) = g(x) \) for every \( x \in D \), then \( f(x) = g(x) \) for all \( x \in S \).

(2) We call a function \( f : S \to \mathbb{R} \) upper semi-continuous if
\[
 f^{-1}((\infty, c)) = f^{-1}(\{y < c\}) 
\]
is open in \( S \) for all \( c \in \mathbb{R} \). Similarly, \( f \) is lower semi-continuous if \( f^{-1}((c, \infty)) \) is open in \( S \) for all \( c \in \mathbb{R} \). Show that \( f \) is upper semi-continuous if and only if
\[
 \limsup_{x \to x_0} f(x) \leq f(x_0) 
\]
for all \( x_0 \in S \); similarly, \( f \) is lower semi-continuous if and only if
\[
 \liminf_{x \to x_0} f(x) \geq f(x_0) 
\]
for all \( x_0 \in S \).

(3) Show that a function \( f : S \to \mathbb{R} \) is continuous if and only if it is both upper and lower semi-continuous.

(4) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function with the property that
\[
 f(x + y) = f(x) + f(y) 
\]
for all \( x, y \in \mathbb{R} \). If \( f(x) \) is continuous at 0, then \( f(x) \equiv cx \) for some constant \( c \in \mathbb{R} \).

(5) We use the notation \( \lfloor x \rfloor \) to denote the largest integer \( \leq x \).
   (a) Find all discontinuities of \( \lfloor cx \rfloor \) for a constant \( c \neq 0 \).
   (b) Show that the infinite series
\[
 \sum_{n=1}^{\infty} \frac{nx - \lfloor nx \rfloor}{2^n} 
\]
converges for all \( x \in \mathbb{R} \).
   (c) Let \( f(x) \) be the function defined by the series in (b). Show that \( f(x) \) is continuous at every \( x \notin \mathbb{Q} \) and discontinuous at every \( x \in \mathbb{Q} \).