Solutions for Math 217 Assignment #1

(1) Find sup$(S)$ and inf$(S)$ for

$$S = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}^+ \right\}.$$ 

Solution. For all $m, n \in \mathbb{Z}^+$, $m, n \geq 1$ and hence

$$\frac{1}{m} + \frac{1}{n} \geq \frac{1}{1} + \frac{1}{1} \geq 2$$

So sup$(S) \leq 2$. And since 2 is achieved when $m = n = 1$, sup$(S) = 2$.

We claim that inf$(S) = 0$. First of all, $\frac{1}{m} + \frac{1}{n} > 0$ for all $m, n \in \mathbb{Z}^+$. So inf$(S) \geq 0$. If $b = \inf(S) > 0$, then $\frac{1}{m} + \frac{1}{n} \geq b$ for all $m, n \in \mathbb{Z}^+$. For $m = n = \frac{2}{b}$, we have

$$\frac{1}{m} + \frac{1}{n} = \frac{2}{n} < b$$

Contradiction. So inf$(S) = 0$.

(2) For two subsets $S_1$ and $S_2$ of $\mathbb{R}$, show that

(a) inf$(S_1 \cup S_2) = \min(\inf(S_1), \inf(S_2))$.
(b) sup$(S_1 \cup S_2) = \max(\sup(S_1), \sup(S_2))$.
(c) inf$(S_1 + S_2) = \inf(S_1) + \inf(S_2)$.
(d) sup$(S_1 + S_2) = \sup(S_1) + \sup(S_2)$.

Here $S_1 + S_2 = \{ x_1 + x_2 : x_1 \in S_1, x_2 \in S_2 \}$ and we allow inf$(S_i) = -\infty$ and sup$(S_i) = \infty$.

Proof. (a) Let $a_1 = \inf(S_1)$, $a_2 = \inf(S_2)$, $a = \min(a_1, a_2)$ and $b = \inf(S_1 \cup S_2)$. For $x \in S_1 \cup S_2$, $x \geq a_1$ if $x \in S_1$ and $x \geq a_2$ if $x \in S_2$. Therefore, $x \geq a$ for all $x \in S_1 \cup S_2$. It follows that $b = \inf(S_1 \cup S_2) \geq a$. On the other hand, $b \leq x$ for all $x \in S_1 \cup S_2$. Therefore, $b \leq a$ for all $x \in S_1$ and hence $b \leq a_1$. Similarly, $b \leq a_2$. Therefore, $b \leq a$. So we may conclude that $a = b$.

(b) We let $-S = \{-x : x \in S\}$. We use the fact that sup$(S) = -\inf(-S)$. Then

$$\sup(S_1 \cup S_2) = -\inf(-S_1 \cup S_2) = -\inf((-S_1) \cup (-S_2))$$

$$= -\min(\inf(S_1), \inf(S_2))$$

$$= -\min(-\sup(S_1), -\sup(S_2))$$

$$= \max(\sup(S_1), \sup(S_2))$$
(c) Let \( a_1 = \inf(S_1) \) and \( a_2 = \inf(S_2) \). Since \( x_1 \geq a_1 \) for all \( x_1 \in S_1 \) and \( x_2 \geq a_2 \) for all \( x_2 \in S_2 \), \( x_1 + x_2 \geq a_1 + a_2 \) for all \( x_1 \in S_1 \) and \( x_2 \in S_2 \). Therefore, \( \inf(S_1 + S_2) \geq a_1 + a_2 \). On the other hand, since \( a_i \) is the infimum of \( S_i \), for all \( b_i > a_i \), there exists \( c_i \in S_i \) such that \( b_i > c_i \) for \( i = 1, 2 \). Therefore,
\[
b_1 + b_2 > c_1 + c_2 \geq \inf(S_1 + S_2)
\]
for all \( b_i > a_i \) and \( i = 1, 2 \). It follows that \( a_1 + a_2 \geq \inf(S_1 + S_2) \).
Hence \( \inf(S_1 + S_2) = \inf(S_1) + \inf(S_2) \).

(d) We have
\[
\sup(S_1 + S_2) = -\inf(-(S_1 + S_2)) = -\inf((-S_1) + (-S_2)) = -\inf(-S_1) - \inf(-S_2) = \sup(S_1) + \sup(S_2)
\]
by (c).

(3) Show that \( \sqrt{3} \notin \mathbb{Q} \).

Proof. Suppose that \( \sqrt{3} \in \mathbb{Q} \). Then \( \sqrt{3} = p/q \) for some integers \( p \) and \( q \) satisfying \( q \neq 0 \) and \( \gcd(p, q) = 1 \). Then \( p^2 = 3q^2 \) and \( 3|p^2 \). Hence \( 3|p \) and \( p = 3m \) for some integer \( m \). Then \( 3m^2 = q^2 \) and \( 3|q^2 \). It follows that \( 3|q \) and we already have \( 3|p \). This contradicts the hypothesis that \( \gcd(p, q) = 1 \).

(4) Let \( I_n \) be a sequence of closed intervals in \( \mathbb{R} \) such that \( I_k \cap I_l \neq \emptyset \) for all \( k, l \in \mathbb{Z}^+ \). Show that
\[
\bigcap_{n=1}^{\infty} I_n \neq \emptyset.
\]

Proof. Let
\[
J_n = \bigcap_{k=1}^{n} I_k
\]
Then
\[
\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} J_n
\]
and \( J_n \supset J_{n+1} \) for all \( n \in \mathbb{Z}^+ \). So by the nested interval theorem, it suffices to show that \( J_n \neq \emptyset \) for all \( n \in \mathbb{Z}^+ \). Let \( I_n = [a_n, b_n] \). Then
\[
I_k \cap I_l = [\max(a_k, a_l), \min(b_k, b_l)].
\]
Since $I_k \cap I_l \neq \emptyset$ for all $k, l \in \mathbb{Z}^+$, $\max(a_k, a_l) \leq \min(b_k, b_l)$ for all $k, l \in \mathbb{Z}^+$ and hence $a_k \leq b_l$ for all $k, l \in \mathbb{Z}^+$. Consequently,

$$\max(a_1, a_2, ..., a_n) \leq \min(b_1, b_2, ..., b_n)$$

for all $n \in \mathbb{Z}^+$. Therefore,

$$J_n = [\max(a_1, a_2, ..., a_n), \min(b_1, b_2, ..., b_n)] \neq \emptyset.$$ 

\[\square\]

(5) Show that $\mathbb{R}(x)$ cannot be made into a complete ordered field, where $\mathbb{R}(x)$ is the field of rational functions in $x$.

Proof. Let $P \subset \mathbb{R}(x)$ be a complete order on $\mathbb{R}(x)$. It suffices to show that there is $z \in \mathbb{R}(x)$ such that $z \not\in \mathbb{R}$ and $0 < z < b$ for all $b \in \mathbb{R}^+$. Suppose that such $z$ exists. Then we let $T = \{az : a \in \mathbb{R}^+\}$. Clearly, $T$ is bounded from above by 1. So $\sup(T) = w$ exists. We have $w \geq az$ for all $\mathbb{R}^+$ and hence $w \geq (a + 1)z$ for all $a \in \mathbb{R}^+$. It follows that $w - z \geq az$ for all $\mathbb{R}^+$ and $w - z$ is an upper bound of $T$. Hence $w - z \geq w$. Contradiction.

We have either $x \in P$ or $-x \in P$. That is, $x > 0$ or $x < 0$. We let $y = x$ if $x > 0$ and $y = -x$ if $x < 0$. Anyway, we have $y > 0$ and $y \not\in \mathbb{R}$. Let $S = \{a \in \mathbb{R}^+ : a > y\}$.

If $S = \emptyset$, then $y > a$ for all $a \in \mathbb{R}^+$ and hence $0 < 1/y < b$ for all $b \in \mathbb{R}^+$. Let $z = 1/y$ and we are done.

If $S \neq \emptyset$, we let $c$ be the infimum of $S$ in $\mathbb{R}$. Note that $\inf(S)$ exists in $\mathbb{R}$ because $\mathbb{R}$ is complete and $S$ is bounded from below by 0. Then we have $c - y > 0$ and $c - y < b$ for all $b \in \mathbb{R}^+$. Let $z = c - y$ and we are done. 

\[\square\]
Solutions for Math 217 Assignment #2

(1) For two functions \( f : A \to B \) and \( g : B \to C \), do the following:

(a) If \( f \) and \( g \) are injective, then \( g \circ f \) is injective.
(b) If \( f \) and \( g \) are surjective, then \( g \circ f \) is surjective.
(c) If \( f \) and \( g \) are bijective, then \( g \circ f \) is bijective.
(d) If \( g \circ f \) is injective, then \( f \) is injective.
(e) If \( g \circ f \) is surjective, then \( g \) is surjective.

**Proof.** (a) For all \( x_1 \neq x_2 \), \( f(x_1) \neq f(x_2) \) since \( f \) is injective. Hence \( g(f(x_1)) \neq g(f(x_2)) \) since \( g \) is injective. Therefore, \( g \circ f \) is injective.

(b) Since \( f \) is surjective, \( f(A) = \{f(x) : x \in A\} = B \). Since \( g \) is surjective, \( g(B) = C \). Therefore, \( g(f(A)) = C \) and \( g \circ f \) is surjective.

(c) By (a) and (b), \( g \circ f \) is both injective and surjective and is hence bijective.

(d) If \( f \) is not injective, \( f(x_1) = f(x_2) \) for some \( x_1 \neq x_2 \in A \). Then \( g(f(x_1)) = g(f(x_2)) \) and hence \( g \circ f \) fails to be injective. Contradiction.

(e) Since \( g \circ f \) is surjective, \( g(f(A)) = C \). And since \( f(A) \subseteq B \), \( g(f(A)) \subseteq g(B) \subseteq C \). Therefore, \( g(B) = C \) and \( g \) is surjective.

(2) Show that a set \( S \) is countable if and only if there exists an injection \( f : S \to \mathbb{Z}^+ \).

**Proof.** If \( S \) is countable, \( S \) is finite or there is a bijection \( f : S \to \mathbb{Z}^+ \). If \( S \) is finite, there is obviously an injection \( f : S \to \mathbb{Z}^+ \). So there is an injection \( f : S \to \mathbb{Z}^+ \) if \( S \) is countable.

Suppose that there is an injection \( f : S \to \mathbb{Z}^+ \). If \( S \) is finite, there is nothing to prove. Suppose that \( S \) is infinite. For \( x \in S \), we let \( T(x) = \{y \in S : f(y) \leq f(x)\} \). We claim that for all \( x_1 \neq x_2 \), we have either \( T(x_1) \subset T(x_2) \) or \( T(x_2) \subset T(x_1) \).

Without the loss of generality, we may assume that \( f(x_1) \leq f(x_2) \). Since \( f \) is injective, \( f(x_1) \neq f(x_2) \). Hence \( f(x_1) < f(x_2) \). Therefore, \( f(y) < f(x_2) \) for all \( y \in T(x_1) \). Hence \( T(x_1) \subset T(x_2) \). And since \( x_2 \not\in T(x_1) \) and \( x_2 \in T(x_2) \). So \( T(x_1) \subset T(x_2) \).

Therefore, for all \( x_1 \neq x_2 \), we have \( |T(x_1)| \neq |T(x_2)| \). Let \( g : S \to \mathbb{Z}^+ \) be the function given by \( g(x) = |T(x)| \). The above shows that \( g \) is injective. We have to show that \( g \) is surjective.
That is, for all \( n \in \mathbb{Z}^+ \), there is \( x \in S \) such that \( |T(x)| = n \). We prove it by induction.

When \( n = 1 \), let \( m = \min f(S) \) and let \( m = f(x) \) for some \( x \in S \). Note that \( S \neq \emptyset \) and hence \( f(S) \neq \emptyset \). Clearly, \( f(y) > f(x) \) for all \( y \neq x \). Therefore, \( T(x) = \{ x \} \) and hence \( |T(x)| = 1 \).

Suppose that there exists \( x \) such that \( |T(x)| = n - 1 \). Let

\[
P = \{ y : y \in S, f(y) > f(x) \}.
\]

Obviously, \( S = P \cup T(x) \). Since \( S \) is infinite, \( P \neq \emptyset \). Let \( m = \min f(P) \) and let \( m = f(x') \) for some \( x' \in S \). So we have \( f(x') > f(x) \) and hence \( T(x) \subset T(x') \). For all \( y \in P \), \( f(y) \geq f(x') \). And since \( f \) is injective, \( f(y) > f(x') \) for all \( y \in P \) and \( y \neq x' \). Therefore,

\[
T(x') = T(x) \cup \{ x' \}
\]

and hence \( |T(x')| = n \). □

(3) Show that a set \( S \) is countable if and only if there exists a surjection \( f : \mathbb{Z}^+ \to S \).

**Proof.** By (2), it is enough to prove that there is an injection \( g : S \to \mathbb{Z}^+ \). For \( x \in S \), we let \( f^{-1}(x) = \{ n \in \mathbb{Z}^+ : f(n) = x \} \).

Since \( f \) is surjective, \( f^{-1}(x) \neq \emptyset \) for all \( x \in S \). We let \( g : S \to \mathbb{Z}^+ \) be the function defined by

\[
g(x) = \min f^{-1}(x).
\]

For all \( x_1 \neq x_2 \), \( f^{-1}(x_1) \cap f^{-1}(x_2) = \emptyset \) and hence \( g(x_1) \neq g(x_2) \). So \( g \) is an injection. □

(4) Show that

\[
||x - y||^2 + ||x + y||^2 = 2(||x||^2 + ||y||^2)
\]

for all \( x, y \in \mathbb{R}^n \).

**Proof.** We have

\[
||x - y||^2 + ||x + y||^2 = \langle x - y, x - y \rangle + \langle x + y, x + y \rangle
\]

\[
= (\langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle) + (\langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle)
\]

\[
= 2(\langle x, x \rangle + \langle y, y \rangle) = 2(||x||^2 + ||y||^2)
\]

□
(5) If $S_1$ and $S_2$ are two convex sets in $\mathbb{R}^n$, then $S_1 + S_2$ is also convex.

Proof. For two points $p_1 + p_2$ and $q_1 + q_2$ in $S_1 + S_2$ with $p_1, q_1 \in S_1$ and $p_2, q_2 \in S_2$ and $0 \leq t \leq 1$,

$$t(p_1 + p_2) + (1 - t)(q_1 + q_2) = (tp_1 + (1 - t)q_1) + (tp_2 + (1 - t)q_2)$$

Since $S_1$ is convex, $tp_1 + (1 - t)q_1 \in S_1$. Since $S_2$ is convex, $tp_2 + (1 - t)q_2 \in S_2$. Therefore, $t(p_1 + p_2) + (1 - t)(q_1 + q_2) \in S_1 + S_2$ and hence $S_1 + S_2$ is convex. \qed