Math 217 Final Review

Materials covered: Sec. 1.1-2.4, 3.1-3.2, 3.4-3.6, 7.1, 8.1-8.2

Some sample problems:

1. Show that the sums of two open sets in $\mathbb{R}^n$ is open. Here the sum $S_1 + S_2$ is defined to be $S_1 + S_2 = \{v_1 + v_2 : v_1 \in S_1, v_2 \in S_2\}$ and we take $S_1 + S_2 = \emptyset$ if one of $S_1$ and $S_2$ is empty.

2. Show that the union of two connected sets in $\mathbb{R}^n$ is connected if they have nonempty intersection.

3. Let $K$ be a nonempty compact set in $\mathbb{R}^2$. Which of the following sets are compact and which are not? Justify your answer.
   - (a) $\{(x, y) : (x + y, x - y) \in K\}$
   - (b) $\{(x, y) : (x, y) \notin K\}$
   - (c) $\{p \in \mathbb{R}^2 : ||p - q|| \leq 1 \text{ for some } q \in K\}$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by
   
   $$f(x) = \begin{cases} 
   1/q & \text{if } x = p/q \text{ for } p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1 \\
   0 & \text{if } x \notin \mathbb{Q}
   \end{cases}$$

   (a) Show that $f(x)$ is not continuous at any $x \in \mathbb{Q}$ and continuous at every $x \notin \mathbb{Q}$.
   (b) Show that $f(x)$ is not differentiable at any $x \notin \mathbb{Q}$.

5. Let $D$ be an open set in $\mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is analytic at $x_0 \in D$ if there is a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $r > 0$ such that

   $$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

   for all $|x - x_0| < r$.

   (a) Show that $f(x)$ is analytic at $x_0$ if there exist positive numbers $R$ and $r$ such that
   
   $$\left| \frac{f^{(n)}(x)}{n!} \right| \leq R^n$$

   for all $|x - x_0| \leq r$.
   (b) Show that $f(x) = 1/(x^3 - x^2)$ is analytic everywhere in $(0, 1)$.

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1http://www.math.ualberta.ca/~xichen/math21710f/fp.pdf
(6) Let \( f(x) \) and \( g(x) \) be two continuous functions on \([a, b]\) with continuous first derivatives \( f'(x) \) and \( g'(x) \) on \((a, b)\).

(a) If \( g'(x) \neq 0 \) for all \( x \in (a, b) \), then there exists a number \( c \in (a, b) \) such that
\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.
\]

(b) Use or not use part (a) to prove the following version of L'Hospital's rule: Suppose that \( f(x_0) = g(x_0) = 0 \) for some \( x_0 \in (a, b) \). Then \( \lim_{x \to x_0} f(x)/g(x) \) exists if \( \lim_{x \to x_0} f'(x)/g'(x) \) exists and
\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.
\]

(7) Show that there does not exist an infinite sequence of real numbers \( a_1, a_2, \ldots, a_n, \ldots \) such that \( \sum_{n=1}^{\infty} a_n^m \) converges and
\[
\sum_{n=1}^{\infty} a_n^m = m
\]
for every positive integer \( m \).

(8) Show that a continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) maps convex sets to convex sets. Is this true for a continuous function \( f : \mathbb{R}^n \to \mathbb{R}^m \) for \( m > 1 \)? Prove it or disprove it by a counterexample.

(9) Let \( a_n \) be a strictly decreasing sequence of real numbers such that \( \lim_{n \to \infty} a_n = 0 \). Show that the series
\[
\sum_{n=1}^{\infty} \left( 1 - \frac{a_{n+1}}{a_n} \right)
\]
diverges.

(10) Let \( S \) be an open and convex set in \( \mathbb{R}^n \) and let \( f : S \to \mathbb{R} \) be a twice continuously partially differentiable function. If \( D_v f(x_0) \geq 0 \) for all \( x_0 \in S \) and \( v \in \mathbb{R}^n \), then
\[
f(x_1) + f(x_2) + \ldots + f(x_m) \geq mf \left( \frac{x_1 + x_2 + \ldots + x_m}{m} \right)
\]
for all \( m \in \mathbb{Z}^+ \) and \( x_1, x_2, \ldots, x_m \in S \), where \( D_v f \) is the directional derivative of \( f \) in the direction of \( v \).