HW 11

Ex. 14.7 | 10. $\cancel{\partial x}\ \dfrac{\partial f}{\partial x} = 6x + 6y - 2$

$$\frac{\partial f}{\partial y} = 6x + 14y + 4$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \implies x = \frac{13}{12},\ y = -\frac{3}{4}$$

$$\frac{\partial^2 f}{\partial x^2} = 6 \qquad \frac{\partial^2 f}{\partial y^2} = 14 \qquad \frac{\partial^2 f}{\partial x \partial y} = 6$$

$$\Delta = f_{xx} f_{yy} - (f_{xy})^2 = \cancel{6\cdot 6\cdot 14}\ 6 \cdot 14 - 6^2 > 0.$$

and $f_{xx} > 0$ when $(x, y) = \left(\frac{13}{12}, -\frac{3}{4}\right)$

$\implies f(x,y)$ has a local minimum at $\left(\frac{13}{12}, -\frac{3}{4}\right)$

20. $\dfrac{\partial f}{\partial x} = -6x + 6y$

$$\frac{\partial f}{\partial y} = 6y - 6y^2 + 6x$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \implies \begin{cases} x = y \\ x + y - \cancel{8}y^2 = 0 \end{cases} \implies \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 2 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = -6, \quad \frac{\partial^2 f}{\partial y^2} = 6 - 12y, \quad \frac{\partial^2 f}{\partial x \partial y} = 6$$

$$\Delta = f_{xx} f_{yy} - f_{xy}^2 = -6(6-12y) - 36$$

At $(x,y)=(0,0)$, $\Delta < 0 \Rightarrow (0,0)$ is a saddle point

At $(x,y)=(2,2)$, $\Delta > 0$, $f_{xx} < 0$

$\Rightarrow (2,2)$ is a local maximum

26. $\frac{\partial f}{\partial x} = 4x^3 + 4y$, $\frac{\partial f}{\partial y} = 4y^3 + 4x$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} x^3 = -y \\ y^3 = -x \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \text{ or } \begin{cases} x=1 \\ y=-1 \end{cases}$$

$$\text{or } \begin{cases} x=-1 \\ y=1 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$\Delta = 144x^2y^2 - 16$$

At $(x,y) = (0,0)$, $\Delta < 0 \Rightarrow (0,0)$ is a saddle point

At $(x,y) = (1,-1)$, $\Delta > 0$, $f_{xx} > 0 \Rightarrow (1,-1)$ is a local maximum

At $(x,y) = (-1,1)$, $\Delta > 0$, $f_{xx} > 0 \Rightarrow (1,-1)$ is a local maximum

Ex. 14.8

4. $\begin{cases} \nabla(x^2y) = \lambda\nabla(x+y-3) \\ x+y=3 \end{cases}$

$\Longrightarrow \begin{cases} 2xy = \lambda \\ x^2 = \lambda \\ x+y=3 \end{cases} \Longrightarrow \begin{cases} x=0 \\ y=3 \end{cases}$ or $\begin{cases} x=2 \\ y=1 \end{cases}$

$f(0,3)=0, \quad f(2,1)=4$

~~$\Rightarrow f_{max}=4$ when $(x,y)=(2,1)$~~

~~$f_{min}=0$ when~~

When $x\to\infty$, $y\to-\infty$, $f(x,y)\to-\infty$

when $x\to-\infty$, $y\to\infty$, $f(x,y)\to\infty$

$\Rightarrow f(x,y)$ has a local maximum at $(2,1)$

a local minimum at $(0,3)$

24. $\begin{cases} \nabla f = \lambda\nabla(x^2+y^2+z^2-25) \\ x^2+y^2+z^2=25 \end{cases}$

$\Longrightarrow \begin{cases} 1 = 2\lambda x \\ 2 = 2\lambda y \\ 3 = 2\lambda z \\ x^2+y^2+z^2=25 \end{cases} \Longrightarrow \begin{cases} x=\frac{1}{2\lambda} \\ y=\frac{1}{\lambda} \\ z=\frac{3}{2\lambda} \\ 4\lambda^2 = 14/25 \end{cases}$

$$\Rightarrow \begin{cases} x = \frac{5}{\sqrt{14}} \\ y = \frac{10}{\sqrt{14}} \\ z = \frac{15}{\sqrt{14}} \end{cases} \quad \text{or} \quad \begin{cases} x = -\frac{5}{\sqrt{14}} \\ y = -\frac{10}{\sqrt{14}} \\ z = -\frac{15}{\sqrt{14}} \end{cases}$$

$$f_{max} = f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = 5\sqrt{14}$$

$$f_{min} = f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -5\sqrt{14}$$

28. Let (x, y, z) be the coordinate of the vertex on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Maximize xyz under the constraints $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $x, y, z > 0$.

$$\begin{cases} \nabla(xyz) = \lambda \nabla\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right) \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} yz = \frac{\lambda}{a} \\ zx = \frac{\lambda}{b} \\ xy = \frac{\lambda}{c} \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \end{cases} \Rightarrow \begin{cases} xyz = \lambda\left(\frac{x}{a}\right) \\ xyz = \lambda\left(\frac{y}{b}\right) \\ xyz = \lambda\left(\frac{z}{c}\right) \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{a}{3} \\ y = \frac{b}{3} \\ z = \frac{c}{3} \end{cases}$$

The largest box has volume $\frac{abc}{27}$