(1) Let

$$f(x) = \cos^2(x^2)$$

(a) Find the Taylor series of $f(x)$ about the point $x = 0$ and its radius of convergence.

(b) Write down an expression to approximate the integral

$$\int_0^{0.1} f(x)dx$$

with an error of magnitude $< 10^{-5}$. You need not simplify your answer but must justify it.

Solution. We have

$$\cos^2(x^2) = \frac{1}{2}(\cos(2x^2) + 1) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x^2)^{2n}$$
$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{4n}$$

for all x. So the radius of convergence is ∞.

Since

$$\int_0^{0.1} f(x)dx = \int_0^{0.1} \left(\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{4n} \right) dx$$
$$= \frac{0.1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}(0.1)^{4n+1}}{(2n)!(4n+1)}$$

$$\left| \int_0^{0.1} f(x)dx - \left(\frac{0.1}{2} + \sum_{k=0}^{n} \frac{(-1)^k 2^{2k-1}(0.1)^{4k+1}}{(2k)!(4k+1)} \right) \right|$$
$$< \frac{2^{2n+1}(0.1)^{4n+5}}{(2n+2)!(4n+5)}$$

When $n = 0$,

$$\frac{2(0.1)^5}{2! \cdot 5} = \frac{10^{-5}}{5} < 10^{-5}$$

Therefore,

$$\int_0^{0.1} f(x)dx \approx \frac{0.1}{2} + \frac{(-1)^0 2^{-1}(0.1)}{0! \cdot 1} = 0.1$$

(2) Find an equation for the plane through points $P = (-1, 1, 0)$, $Q = (8, -3, -1)$ and $R = (-4, 1, 1)$ and the area of the triangle ΔPQR.

Solution. We have

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (9i - 4j - k) \times (-3i + k)$$
$$= \det \begin{bmatrix} i & j & k \\ 9 & -4 & -1 \\ -3 & 0 & 1 \end{bmatrix} = -4i - 6j - 12k$$

Therefore, the plane through PQR is

$$-4(x+1) - 6(y-1) - 12z = 0 \Leftrightarrow 2x + 3y + 6z = 1$$

and the area of the triangle PQR is

$$S_{\Delta PQR} = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}| - 4i - 6j - 12k| = 7$$

(3) Let $w = f(x,y)$, $x = r\cos\theta$, and $y = r\sin\theta$. Find $\partial w/\partial r$ and $\partial w/\partial\theta$ and express your answers in terms of r, θ, f_x and f_y.
Solution. By chain rule,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

and

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$$

(4) Let

$$f(x,y) = x^4 + y^4 + 4xy$$

(a) Find all the critical points of $f(x,y)$ and identify local maxima, local minima and saddle points among them.
(b) Find the absolute maximum and minimum of $f(x,y)$ in the region given by $-2 \le x \le 2$ and $-2 \le y \le 2$.

Solution. (a) Solve $f_x = f_y = 0$, i.e., $4x^3 + 4y = 4y^3 + 4x = 0 \Rightarrow x^9 - x = 0 \Rightarrow x = 0, \pm 1$. Therefore, there are three critical points $(0,0)$, $(1,-1)$ and $(-1,1)$. Note that $f_{xx} = 12x^2$, $f_{xy} = 4$, $f_{yy} = 12y^2$ and $\Delta = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$. At $(x,y) = (0,0)$, $\Delta < 0$ and hence $(0,0)$ is a saddle point. At $(x,y) = (1,-1)$ and $(x,y) = (-1,1)$, $\Delta > 0$ and $f_{xx} > 0$; hence $(1,-1)$ and $(-1,1)$ are local minima.
(b) At the local maxima and minima, $f(1,-1) = f(-1,1) = -2$.

When restricted to $x = -2$, $g(y) = f(-2,y) = y^4 - 8y + 16$ for $-2 \le y \le 2$. Solve $g'(y) = 0$, i.e., $4y^3 - 8 = 0 \Rightarrow y = \sqrt[3]{2}$.

When restricted to $x = 2$, $g(y) = f(2,y) = y^4 + 8y + 16$ for $-2 \le y \le 2$. Solve $g'(y) = 0$, i.e., $4y^3 + 8 = 0 \Rightarrow y = -\sqrt[3]{2}$.

When restricted to $y=-2$, $g(x)=f(x,-2)=x^4-8x+16$ for $-2\le x\le 2$. Solve $g'(x)=0$, i.e., $4x^3-8=0 \Rightarrow x=\sqrt[3]{2}$.

When restricted to $y=2$, $g(x)=f(x,2)=x^4+8x+16$ for $-2\le x\le 2$. Solve $g'(x)=0$, i.e., $4x^3+8=0 \Rightarrow x=-\sqrt[3]{2}$.

We compute

$$
\begin{aligned}
f(1,-1) &= f(-1,1) = -2 \\
f(-2,-2) &= f(2,2) = 48, f(-2,2) = f(2,-2) = 16 \\
f(-2,\sqrt[3]{2}) &= f(2,-\sqrt[3]{2}) = f(\sqrt[3]{2},-2) = f(-\sqrt[3]{2},2) = 16-6\sqrt[3]{2}
\end{aligned}
$$

Therefore, $f_{\max}=48$ when $(x,y)=(-2,-2)$ or $(2,2)$ and $f_{\min}=-2$ when $(x,y)=(1,-1)$ or $(-1,1)$.

(5) Find an equation of the plane through the point $(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x+y-z=3$ and $x+2y+z=2$.

Solution. Since $\mathbf{u}=\langle 2,1,-1\rangle$ and $\mathbf{v}=\langle 1,2,1\rangle$ are the vectors perpendicular to the planes, the vector

$$
\mathbf{u}\times\mathbf{v} = \det\begin{bmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = 3i-3j+3k
$$

is in the same direction of the line of intersection. Therefore, the plane through $(2,1,-1)$ and perpendicular to the line of intersection is

$$
3(x-2)-3(y-1)+3(z+1)=0 \Leftrightarrow x-y+z=0
$$

(6) Find the points on the surface

$$
xy+yz+zx-x-z^2=0
$$

where the tangent plane is parallel to the xy-plane.

Solution. The gradient of $f(x,y,z)=xy+yz+zx-x-z^2$ is $\nabla f=\langle y+z-1, x+z, y+x-2z\rangle$. The tangent plane is parallel to xy-plane if ∇f is in the same direction of the vector $\langle 0,0,1\rangle$, i.e., when $y+z-1=x+z=0$. We solve the system of equations

$$
\begin{cases} xy+yz+zx-x-z^2=0 \\ y+z-1=0 \\ x+z=0 \end{cases} \Leftrightarrow \begin{cases} -x-2x^2=0 \\ y+z-1=0 \\ x+z=0 \end{cases}
$$

$$
\Leftrightarrow \begin{cases} x=0 \\ y=1 \\ z=0 \end{cases} \text{or} \begin{cases} x=-1/2 \\ y=1/2 \\ z=1/2 \end{cases}
$$

Therefore, the tangent planes of the surface at $(0,1,0)$ and $(-1/2,1/2,1/2)$ are parallel to the xy-plane.

(7) Find the point closest to the origin on the curve of intersection of the plane $2y+4z=5$ and the cone $z^2=4x^2+4y^2$.

Solution. Let $P=(x,y,z)$ be a point on the curve. We are minimize $f(x,y,z)=|OP|^2=x^2+y^2+z^2$ under the constraints $g_1(x,y,z)=2y+4z-5=0$ and $g_2(x,y,z)=4x^2+4y^2-z^2=0$. Using Lagrange multipliers, we solve the system of equations

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = g_2 = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = 8\lambda_2 x \\ 2y = 2\lambda_1 + 8\lambda_2 y \\ 2z = 4\lambda_1 - 2\lambda_2 z \\ 0 = 2y+4z-5 \\ 0 = 4x^2+4y^2-z^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ 2y = 2\lambda_1 + 8\lambda_2 y \\ 2z = 4\lambda_1 - 2\lambda_2 z \\ 0 = 2y+4z-5 \\ 0 = 4x^2+4y^2-z^2 \end{cases} \text{or} \begin{cases} \lambda_2 = 1/4 \\ 2y = 2\lambda_1 + 8\lambda_2 y \\ 2z = 4\lambda_1 - 2\lambda_2 z \\ 0 = 2y+4z-5 \\ 0 = 4x^2+4y^2-z^2 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ 0 = 2y+4z-5 \\ 0 = 4y^2-z^2 \end{cases} \text{or} \begin{cases} \lambda_2 = 1/4 \\ \lambda_1 = 0 \\ z = 0 \\ y = 5/2 \\ 0 = 4x^2+25 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 1/2 \\ z = 1 \end{cases} \text{or} \begin{cases} x = 0 \\ y = -5/6 \\ z = 5/3 \end{cases}$$

Since $f(0,1/2,1)=5/4$ and $f(0,-5/6,5/3)=125/36>5/4$, $(0,1/2,1)$ is the closest point to the origin on the curve.

(8) Let C be a plane curve given by

$$\mathbf{s}(t) = (2\ln t)\mathbf{i} - \left(t + \frac{1}{t}\right)\mathbf{j}$$

(a) Find the length of C for $1 \le t \le 2$.
(b) Find the curvature of C at $t=1$.
(c) Find the osculating circle to the curve C at $t=1$.

Solution. (a) We have

$$\mathbf{v} = \frac{d\mathbf{s}}{dt} = \frac{2}{t}\mathbf{i} - \left(1 - \frac{1}{t^2}\right)\mathbf{j} \text{ and } |\mathbf{v}| = 1 + \frac{1}{t^2}$$

Then the length of C for $1 \le t \le 2$ is

$$\int_1^2 |\mathbf{v}| dt = \int_1^2 \left(1 + \frac{1}{t^2}\right) dt = \left(t - \frac{1}{t}\right)\Bigg|_1^2 = \frac{3}{2}$$

(b) Since

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2t}{1+t^2}\mathbf{i} - \frac{t^2-1}{t^2+1}\mathbf{j}$$

we have

$$\frac{d\mathbf{T}}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\mathbf{i} - \frac{4t}{(1+t^2)^2}\mathbf{j}$$

Hence

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{2t^2}{(1+t^2)^2}$$

and $\kappa = 1/2$ when $t = 1$.
(c) Let $P = (0, -2)$ and let $Q = (a, b)$ be the center of the osculating circle. Then the osculating circle at P is

$$(x-a)^2 + (y-b)^2 = 4$$

and hence

$$a^2 + (b+2)^2 = 4$$

On the other hand, we have

$$\overrightarrow{PQ} = \lambda \left.\frac{d\mathbf{T}}{dt}\right|_{t=1} \Rightarrow a\mathbf{i} + (b+2)\mathbf{j} = -\lambda\mathbf{j}$$

for some $\lambda > 0$. Therefore, $a = 0$ and $(b+2)^2 = 4$. Since $\lambda > 0$, $b+2 < 0$. Therefore, $b+2 = -2$ and $b = -4$. So the osculating circle is

$$x^2 + (y+4)^2 = 4$$