

- (1) Find the point of intersection of the lines  $x = t + 2, y = 3t + 4, z = 4t + 5$ , and  $x = 6s + 13, y = 5s + 11, z = 4s + 9$ , and then find the plane containing these two lines.

**Solution.** Solve the system of equations

$$\begin{cases} t + 2 = 6s + 13 \\ 3t + 4 = 5s + 11 \end{cases}$$

and we obtain  $t = -1$  and  $s = -2$ . So the two lines meet at the point  $(1, 1, 1)$ . The direction of the two lines is given by the vector  $\mathbf{u} = i + 3j + 4k$  and  $\mathbf{v} = 6i + 5j + 4k$ . The cross product is

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} i & j & k \\ 1 & 3 & 4 \\ 6 & 5 & 4 \end{bmatrix} = -8i + 20j - 13k$$

So the plane containing the two lines is

$$-8(x - 1) + 20(y - 1) - 13(z - 1) = 0 \Leftrightarrow 8x - 20y + 13z = 1$$

- (2) Let  $w = f(u, v)$ ,  $u = x + y$ , and  $v = xy$ . Find  $\partial w / \partial x$  and  $\partial w / \partial y$  and express your answers in terms of  $x, y, \partial w / \partial u$  and  $\partial w / \partial v$ .

**Solution.** By chain rule,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u} + x \frac{\partial w}{\partial v}$$

- (3) Let

$$f(x, y) = (4x - x^2) \cos y$$

- (a) Find all the critical points of  $f(x, y)$  and identify local maxima, local minima and saddle points among them.  
 (b) Find the absolute maximum and minimum of  $f(x, y)$  in the region given by  $1 \leq x \leq 3$  and  $-\pi/4 \leq y \leq \pi/4$ .

**Solution.** (a) Solve the equation  $\partial f / \partial x = \partial f / \partial y = 0$ :

$$\begin{cases} (4 - 2x) \cos y = 0 \\ -(4x - x^2) \sin y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 2 \text{ or } y = k\pi + \frac{\pi}{2} \\ x = 0, 4 \text{ or } y = k\pi \end{cases}$$

So the critical points are  $(2, k\pi)$ ,  $(0, k\pi + \pi/2)$  and  $(4, k\pi + \pi/2)$  for  $k$  integers. The discriminant is

$$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 2(4x - x^2) \cos^2 y - (4 - 2x)^2 \sin^2 y$$

Since

$$\Delta|_{(2,k\pi)} = 8 > 0 \text{ and } f_{xx}(2, k\pi) = -2 \cos(k\pi) = -2(-1)^k$$

$(2, k\pi)$  is a local maximum if  $k$  is even and  $(2, k\pi)$  is a local minimum if  $k$  is odd. Since

$$\Delta|_{(0,k\pi+\pi/2)} = \Delta|_{(4,k\pi+\pi/2)} = -16 < 0$$

$(0, k\pi + \pi/2)$  and  $(4, k\pi + \pi/2)$  are saddle points.

(b) The function has only one critical point  $(2, 0)$  in the region. Let us find the absolute maximum/minimum when we restrict it to the boundary of the region. When we restrict it to  $x = 1$ ,  $f(1, y) = g(y) = 3 \cos y$  has a critical point at  $y = 0$ . When we restrict it to  $x = 3$ ,  $f(3, y) = 3 \cos y$  has a critical point at  $y = 0$ . When we restrict it to  $y = -\pi/4$ ,  $f(x, -\pi/4) = g(x) = (\sqrt{2}/2)(4x - x^2)$  has a critical point at  $x = 2$ . When we restrict it to  $y = \pi/4$ ,  $f(x, \pi/4) = g(x) = (\sqrt{2}/2)(4x - x^2)$  has a critical point at  $x = 2$ . Therefore, we compute

$$f(2, 0) = 4,$$

$$f(1, -\pi/4) = f(3, -\pi/4) = f(1, \pi/4) = f(3, \pi/4) = \frac{3}{2}\sqrt{2},$$

$$f(1, 0) = f(3, 0) = 3, \text{ and } f(2, -\pi/4) = f(2, \pi/4) = 2\sqrt{2}$$

Therefore,  $f(x, y)$  achieves the maximum 4 at  $(2, 0)$  and the minimum  $3\sqrt{2}/2$  at  $(1, \pm\pi/4)$  and  $(3, \pm\pi/4)$ .

- (4) Find an equation for the plane through points  $P = (-1, 1, 0)$ ,  $Q = (8, -3, -1)$  and  $R = (-4, 1, 1)$  and the area of the triangle  $\Delta PQR$ .

**Solution.** We have

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= (9i - 4j - k) \times (-3i + k) \\ &= \det \begin{bmatrix} i & j & k \\ 9 & -4 & -1 \\ -3 & 0 & 1 \end{bmatrix} = -4i - 6j - 12k \end{aligned}$$

Therefore, the plane through  $PQR$  is

$$-4(x + 1) - 6(y - 1) - 12z = 0 \Leftrightarrow 2x + 3y + 6z = 1$$

and the area of the triangle  $PQR$  is

$$S_{\Delta PQR} = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} |-4i - 6j - 12k| = 7$$

(5) Let

$$f(x) = \frac{1}{(x+1)(x+2)(x+3)}$$

- (a) Find the Taylor series of  $f(x)$  about the point  $x = 0$  and its radius of convergence.  
 (b) Compute the value of

$$\sum_{n=0}^{\infty} f(n)$$

**Solution.** (a) Let us try to write  $f(x)$  as a sum of partial fractions:

$$\frac{1}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$\Rightarrow 1 = A(x+2)(x+3) + B(x+3)(x+1) + C(x+1)(x+2)$$

Setting  $x = -1$  yields  $A = 1/2$ ; setting  $x = -2$  yields  $B = -1$ ; setting  $x = -3$  yields  $C = 1/2$ . Therefore,

$$f(x) = \frac{1}{2} \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{2} \frac{1}{x+3}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} x^n + \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n} x^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left( 1 - \frac{1}{2^n} + \frac{1}{3^{n+1}} \right) x^n = \sum_{n=0}^{\infty} a_n x^n$$

The radius of convergence of the above Taylor series is

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1/2)(-1)^n(1 - 2^{-n} + 3^{-n-1})}{(1/2)(-1)^{n+1}(1 - 2^{-n-1} + 3^{-n-2})} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{1 - 2^{-n} + 3^{-n-1}}{1 - 2^{-n-1} + 3^{-n-2}} = 1$$

(b)

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \left( \left( \frac{1}{n+1} - \frac{1}{n+2} \right) - \left( \frac{1}{n+2} - \frac{1}{n+3} \right) \right)$$

$$= \frac{1}{2} \left( 1 - \frac{1}{2} \right) = \frac{1}{4}$$

(6) Let

$$f(x, y, z) = x^3 + x^2y + y^2z$$

- (a) Find the gradient  $\nabla f$  of  $f$ .  
 (b) Suppose that the surface  $f(x, y, z) = c$  passes through the point  $(1, 2, 1)$ . Find the constant  $c$  and the equation of the tangent plane to the surface at  $(1, 2, 1)$ .

**Solution.** (a)

$$\nabla f = f_x i + f_y j + f_z k = (3x^2 + 2xy)i + (x^2 + 2yz)j + y^2 k$$

- (b) Obviously,  $c = f(1, 2, 1) = 7$ . The tangent plane is orthogonal to

$$\nabla f|_{(1,2,1)} = 7i + 5j + 4k$$

and hence it is

$$7(x - 1) + 5(y - 2) + 4(z - 1) = 0 \Leftrightarrow 7x + 5y + 4z = 21$$

- (7) (a) Maximize  $f(x, y, z) = xyz$  subject to the constraints  $x^2 + y^2 + z^2 = 1$ .  
 (b) Find the dimensions of the rectangular box with the largest volume that can be inscribed in the unit sphere.

**Solution.** (a) By Lagrange multiplier,  $f(x, y, z)$  achieves the maximum when

$$\begin{aligned} \begin{cases} \nabla f = \lambda \nabla(x^2 + y^2 + z^2 - 1) \\ 1 = x^2 + y^2 + z^2 \end{cases} &\Leftrightarrow \begin{cases} xy = 2\lambda z \\ yz = 2\lambda x \\ zx = 2\lambda y \\ 1 = x^2 + y^2 + z^2 \end{cases} \\ \Rightarrow \begin{cases} \lambda x^2 = \lambda y^2 = \lambda z^2 \\ 1 = x^2 + y^2 + z^2 \end{cases} \end{aligned}$$

So we either have  $\lambda = xy = yz = zx = 0$  or  $x^2 = y^2 = z^2 = 1/3$ . So the critical points are  $(\pm 1, 0, 0)$ ,  $(0, \pm 1, 0)$ ,  $(0, 0, \pm 1)$  and  $(\pm\sqrt{3}/3, \pm\sqrt{3}/3, \pm\sqrt{3}/3)$ ;  $f(x, y, z)$  achieves the maximum  $\sqrt{3}/9$  when  $|x| = |y| = |z| = \sqrt{3}/3$  and  $xyz > 0$ .

(b) We set up the  $xyz$  coordinates such that the unit sphere is given by  $x^2 + y^2 + z^2 = 1$  and the faces of the rectangular box are parallel to the axes. Then the coordinates of the vertices of the box are given by  $(\pm a, \pm b, \pm c)$  with  $a^2 + b^2 + c^2 = 1$  and  $a, b, c > 0$ . The box has dimension  $2a \times 2b \times 2c$  and hence has volume  $f(a, b, c) = 8abc$ . So it suffices to maximize  $f(a, b, c)$  under the constraint  $a^2 + b^2 + c^2 = 1$ . We have already solved

this problem in part (a). So the largest box is a cube with dimension  $(2\sqrt{3}/3) \times (2\sqrt{3}/3) \times (2\sqrt{3}/3)$ .

(8) Let  $C$  be a plane curve given by

$$\mathbf{s}(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^2}{2}\mathbf{j}$$

- (a) Find the length of  $C$  for  $1 \leq t \leq 2$ .  
 (b) Find the curvature of  $C$  at  $t = 1$ .  
 (c) Find the osculating circle to the curve  $C$  at  $t = 1$ .

**Solution.** (a) The length is

$$\begin{aligned} \int_1^2 |\mathbf{v}| dt &= \int_1^2 \left| \frac{d\mathbf{s}}{dt} \right| dt \\ &= \int_1^2 |t^2\mathbf{i} + t\mathbf{j}| dt = \int_1^2 t\sqrt{1+t^2} dt \\ &= \frac{1}{2} \int_1^2 \sqrt{1+t^2} d(t^2) = \frac{1}{2} \int_1^4 \sqrt{1+u} du \\ &= \frac{1}{3} (1+u)^{3/2} \Big|_1^4 = \frac{1}{3} (5\sqrt{5} - 2\sqrt{2}) \end{aligned}$$

(b) The unit tangent vector is given by

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{t^2\mathbf{i} + t\mathbf{j}}{t\sqrt{1+t^2}} = \frac{t}{\sqrt{1+t^2}}\mathbf{i} + \frac{1}{\sqrt{1+t^2}}\mathbf{j} \end{aligned}$$

So the normal vector is

$$\begin{aligned} \frac{d\mathbf{T}}{dt} &= \left( \frac{t}{\sqrt{1+t^2}} \right)' \mathbf{i} + \left( \frac{1}{\sqrt{1+t^2}} \right)' \mathbf{j} \\ &= \left( \frac{1}{\sqrt{1+t^2}} - \frac{t^2}{(1+t^2)\sqrt{1+t^2}} \right) \mathbf{i} - \frac{t}{(1+t^2)\sqrt{1+t^2}} \mathbf{j} \\ &= \frac{1}{(1+t^2)\sqrt{1+t^2}} \mathbf{i} - \frac{t}{(1+t^2)\sqrt{1+t^2}} \mathbf{j} \end{aligned}$$

and the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{t(1+t^2)\sqrt{1+t^2}}$$

At  $t = 1$ ,  $\kappa = \sqrt{2}/4$ .

(c) Let  $P = \mathbf{s}(1) = (1/3, 1/2)$  and let  $Q = (a, b)$  be the center of the osculating circle. The radius of the osculating circle is

$1/\kappa = 2\sqrt{2}$ . So the circle is given by

$$(x - a)^2 + (y - b)^2 = 8$$

Since the circle passes through  $P$ , we have

$$\left(\frac{1}{3} - a\right)^2 + \left(\frac{1}{2} - b\right)^2 = 8$$

And the vector  $\overrightarrow{PQ}$  is in the same direction of the normal vector  $d\mathbf{T}/dt$ . Therefore,

$$\left(a - \frac{1}{3}\right)\mathbf{i} + \left(b - \frac{1}{2}\right)\mathbf{j} = \lambda(\mathbf{i} - \mathbf{j})$$

for some  $\lambda > 0$ . Therefore,

$$a - \frac{1}{3} = \lambda, b - \frac{1}{2} = -\lambda \text{ and } 2\lambda^2 = 8$$

Since  $\lambda > 0$ ,  $\lambda = 2$ ,  $a = 7/3$  and  $b = -3/2$ . The osculating circle is

$$\left(x - \frac{7}{3}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 8$$