(1) Find the point of intersection of the lines $x = t + 2, y = 3t + 4, z = 4t + 5$, and $x = 6s + 13, y = 5s + 11, z = 4s + 9$, and then find the plane containing these two lines.

**Solution.** Solve the system of equations

$$\begin{cases} 
  t + 2 = 6s + 13 \\
  3t + 4 = 5s + 11 
\end{cases}$$

and we obtain $t = -1$ and $s = -2$. So the two lines meet at the point $(1, 1, 1)$. The direction of the two lines is given by the vector $\mathbf{u} = i + 3j + 4k$ and $\mathbf{v} = 6i + 5j + 4k$. The cross product is

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} i & j & k \\ 1 & 3 & 4 \\ 6 & 5 & 4 \end{bmatrix} = -8i + 20j - 13k$$

So the plane containing the two lines is

$$-8(x - 1) + 20(y - 1) - 13(z - 1) = 0 \iff 8x - 20y + 13z = 1$$

(2) Let $w = f(u, v), u = x + y, \text{ and } v = xy$. Find $\partial w/\partial x$ and $\partial w/\partial y$ and express your answers in terms of $x, y, \partial w/\partial u$ and $\partial w/\partial v$.

**Solution.** By chain rule,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

(3) Let

$$f(x, y) = (4x - x^2) \cos y$$

(a) Find all the critical points of $f(x, y)$ and identify local maxima, local minima and saddle points among them.

(b) Find the absolute maximum and minimum of $f(x, y)$ in the region given by $1 \leq x \leq 3 \text{ and } -\pi/4 \leq y \leq \pi/4$.

**Solution.** (a) Solve the equation $\partial f/\partial x = \partial f/\partial y = 0$:

$$\begin{cases} 
  (4 - 2x) \cos y = 0 \\
  -(4x - x^2) \sin y = 0 
\end{cases} \iff \begin{cases} 
  x = 2 \text{ or } y = k\pi + \frac{\pi}{2} \\
  x = 0, 4 \text{ or } y = k\pi 
\end{cases}$$

So the critical points are $(2, k\pi), (0, k\pi + \pi/2)$ and $(4, k\pi + \pi/2)$ for $k$ integers. The discriminant is

$$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 2(4x - x^2)\cos^2 y - (4 - 2x)^2 \sin^2 y$$
Since
$$\Delta|_{(2,k\pi)} = 8 > 0 \text{ and } f_{xx}(2,k\pi) = -2\cos(k\pi) = -2(-1)^k$$

$(2,k\pi)$ is a local maximum if $k$ is even and $(2,k\pi)$ is a local minimum if $k$ is odd. Since
$$\Delta|_{(0,k\pi+\pi/2)} = \Delta|_{(4,k\pi+\pi/2)} = -16 < 0$$

$(0,k\pi+\pi/2)$ and $(4,k\pi+\pi/2)$ are saddle points.

(b) The function has only one critical point $(2,0)$ in the region. Let us find the absolute maximum/minimum when we restrict it to the boundary of the region. When we restrict it to $x = 1$, $f(1,y) = g(y) = 3\cos y$ has a critical point at $y = 0$. When we restrict it to $x = 3$, $f(3,y) = 3\cos y$ has a critical point at $y = 0$. When we restrict it to $y = -\pi/4$, $f(x,-\pi/4) = g(x) = (\sqrt{2}/2)(4x-x^2)$ has a critical point at $x = 2$. When we restrict it to $y = \pi/4$, $f(x,\pi/4) = g(x) = (\sqrt{2}/2)(4x-x^2)$ has a critical point at $x = 2$. Therefore, we compute
$$f(2,0) = 4,$$

$$f(1,-\pi/4) = f(3,-\pi/4) = f(1,\pi/4) = f(3,\pi/4) = \frac{3}{2}\sqrt{2},$$

$$f(1,0) = f(3,0) = 3, \text{ and } f(2,-\pi/4) = f(2,\pi/4) = 2\sqrt{2}$$

Therefore, $f(x,y)$ achieves the maximum 4 at $(2,0)$ and the minimum $3\sqrt{2}/2$ at $(1,\pm\pi/4)$ and $(3,\pm\pi/4)$.

(4) Find an equation for the plane through points $P = (-1,1,0)$, $Q = (8,-3,-1)$ and $R = (-4,1,1)$ and the area of the triangle $\Delta PQR$.

Solution. We have

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (9i - 4j - k) \times (-3i + k)$$

$$= \det \begin{bmatrix} i & j & k \\ 9 & -4 & -1 \\ -3 & 0 & 1 \end{bmatrix} = -4i - 6j - 12k$$

Therefore, the plane through $PQR$ is

$$-4(x+1) - 6(y-1) - 12z = 0 \iff 2x + 3y + 6z = 1$$

and the area of the triangle $PQR$ is

$$S_{\Delta PQR} = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}|-4i - 6j - 12k| = 7$$
(5) Let
\[ f(x) = \frac{1}{(x + 1)(x + 2)(x + 3)} \]

(a) Find the Taylor series of \( f(x) \) about the point \( x = 0 \) and its radius of convergence.
(b) Compute the value of
\[ \sum_{n=0}^{\infty} f(n) \]

**Solution.**

(a) Let us try to write \( f(x) \) as a sum of partial fractions:
\[ \frac{1}{(x + 1)(x + 2)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x + 2} + \frac{C}{x + 3} \]
\[ \Rightarrow 1 = A(x + 2)(x + 3) + B(x + 3)(x + 1) + C(x + 1)(x + 2) \]

Setting \( x = -1 \) yields \( A = 1/2 \); setting \( x = -2 \) yields \( B = -1 \); setting \( x = -3 \) yields \( C = 1/2 \). Therefore,
\[ f(x) = \frac{1}{2} \frac{1}{x + 1} - \frac{1}{2} \frac{1}{x + 2} + \frac{1}{2} \frac{1}{x + 3} \]
\[ = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} x^n + \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n} x^n \]
\[ = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left( 1 - \frac{1}{2^n} + \frac{1}{3^{n+1}} \right) x^n = \sum_{n=0}^{\infty} a_n x^n \]

The radius of convergence of the above Taylor series is
\[ \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(1/2)(-1)^n(1 - 2^{-n} + 3^{-n-1})}{(1/2)(-1)^{n+1}(1 - 2^{-n-1} + 3^{-n-2})} \right| \]
\[ = \lim_{n \to \infty} \frac{1 - 2^{-n} + 3^{-n-1}}{1 - 2^{-n-1} + 3^{-n-2}} = 1 \]

(b)
\[ \sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{n + 1} - \frac{2}{n + 2} + \frac{1}{n + 3} \right) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{2} \left( \left( \frac{1}{n + 1} - \frac{1}{n + 2} \right) - \left( \frac{1}{n + 2} - \frac{1}{n + 3} \right) \right) \]
\[ = \frac{1}{2} \left( 1 - \frac{1}{2} \right) = \frac{1}{4} \]
(6) Let 
\[ f(x, y, z) = x^3 + x^2 y + y^2 z \]
(a) Find the gradient \( \nabla f \) of \( f \).
(b) Suppose that the surface \( f(x, y, z) = c \) passes through the point \((1, 2, 1)\). Find the constant \( c \) and the equation of the tangent plane to the surface at \((1, 2, 1)\).

\[ \text{Solution.} \quad (a) \quad \nabla f = f_x i + f_y j + f_z k = (3x^2 + 2xy)i + (x^2 + 2yz)j + y^2k \]

(b) Obviously, \( c = f(1, 2, 1) = 7 \). The tangent plane is orthogonal to \( \nabla f \big|_{(1,2,1)} = 7i + 5j + 4k \) and hence it is
\[ 7(x - 1) + 5(y - 2) + 4(z - 1) = 0 \iff 7x + 5y + 4z = 21 \]

(7) (a) Maximize \( f(x, y, z) = xyz \) subject to the constraints \( x^2 + y^2 + z^2 = 1 \).
(b) Find the dimensions of the rectangular box with the largest volume that can be inscribed in the unit sphere.

\[ \text{Solution.} \quad (a) \quad \text{By Lagrange multiplier, } f(x, y, z) \text{ achieves the maximum when } \]
\[ \begin{align*}
\nabla f &= \lambda \nabla (x^2 + y^2 + z^2 - 1) \\
1 &= x^2 + y^2 + z^2
\end{align*} \quad \Rightarrow \quad \begin{cases}
xy = 2\lambda z \\
yz = 2\lambda x \\
zx = 2\lambda y \\
1 = x^2 + y^2 + z^2
\end{cases} \]
\[ \Rightarrow \begin{cases}
\lambda x^2 = \lambda y^2 = \lambda z^2 \\
1 = x^2 + y^2 + z^2
\end{cases} \]

So we either have \( \lambda = xy = yz = zx = 0 \) or \( x^2 = y^2 = z^2 = 1/3 \). So the critical points are \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\) and \((\pm \sqrt{3}/3, \pm \sqrt{3}/3, \pm \sqrt{3}/3)\); \( f(x, y, z) \) achieves the maximum \( \sqrt{3}/9 \) when \( |x| = |y| = |z| = \sqrt{3}/3 \) and \( xyz > 0 \).
(b) We set up the \( xyz \) coordinates such that the unit sphere is given by \( x^2 + y^2 + z^2 = 1 \) and the faces of the rectangular box are parallel to the axises. Then the coordinates of the vertices of the box are given by \((\pm a, \pm b, \pm c)\) with \( a^2 + b^2 + c^2 = 1 \) and \( a, b, c > 0 \). The box has dimension \( 2a \times 2b \times 2c \) and hence has volume \( f(a, b, c) = 8abc \). So it suffices to maximize \( f(a, b, c) \) under the constraint \( a^2 + b^2 + c^2 = 1 \). We have already solved
this problem in part (a). So the largest box is a cube with dimension \((2\sqrt{3}/3) \times (2\sqrt{3}/3) \times (2\sqrt{3}/3)\).

(8) Let \(C\) be a plane curve given by

\[
s(t) = \frac{t^3}{3}i + \frac{t^2}{2}j
\]

(a) Find the length of \(C\) for \(1 \leq t \leq 2\).
(b) Find the curvature of \(C\) at \(t = 1\).
(c) Find the osculating circle to the curve \(C\) at \(t = 1\).

**Solution.** (a) The length is

\[
\int_1^2 |v| dt = \int_1^2 \left| \frac{ds}{dt} \right| dt
\]

\[
= \int_1^2 |t^2i + tj| dt = \int_1^2 t\sqrt{1 + t^2} dt
\]

\[
= \frac{1}{2} \int_1^2 \sqrt{1 + t^2} d(t^2) = \frac{1}{2} \int_1^4 \sqrt{1 + u} du
\]

\[
= \frac{1}{3} (1 + u)^{3/2} \bigg|_1^4 = \frac{1}{3} (5\sqrt{5} - 2\sqrt{2})
\]

(b) The unit tangent vector is given by

\[
T = \frac{v}{|v|}
\]

\[
= \frac{t^2i + tj}{t\sqrt{1 + t^2}} = \frac{t}{\sqrt{1 + t^2}}i + \frac{1}{\sqrt{1 + t^2}}j
\]

So the normal vector is

\[
\frac{dT}{dt} = \left( \frac{t}{\sqrt{1 + t^2}} \right)'i + \left( \frac{1}{\sqrt{1 + t^2}} \right)'j
\]

\[
= \left( \frac{1}{\sqrt{1 + t^2}} - \frac{t^2}{(1 + t^2)^2\sqrt{1 + t^2}} \right)i - \frac{t}{(1 + t^2)^2\sqrt{1 + t^2}}j
\]

\[
= \frac{1}{(1 + t^2)^{3/2}}i - \frac{t}{(1 + t^2)^{3/2}}j
\]

and the curvature is

\[
\kappa = \frac{1}{|v|} \left| \frac{dT}{dt} \right| = \frac{1}{t(1 + t^2)^{3/2}}
\]

At \(t = 1\), \(\kappa = \sqrt{2}/4\).

(c) Let \(P = s(1) = (1/3, 1/2)\) and let \(Q = (a, b)\) be the center of the osculating circle. The radius of the osculating circle is
$1/\kappa = 2\sqrt{2}$. So the circle is given by
\[
(x - a)^2 + (y - b)^2 = 8
\]
Since the circle passes through $P$, we have
\[
\left(\frac{1}{3} - a\right)^2 + \left(\frac{1}{2} - b\right)^2 = 8
\]
And the vector $\overrightarrow{PQ}$ is in the same direction of the normal vector $d\mathbf{T}/dt$. Therefore,
\[
\left( a - \frac{1}{3}\right)\mathbf{i} + \left( b - \frac{1}{2}\right)\mathbf{j} = \lambda (\mathbf{i} - \mathbf{j})
\]
for some $\lambda > 0$. Therefore,
\[
a - \frac{1}{3} = \lambda, b - \frac{1}{2} = -\lambda \text{ and } 2\lambda^2 = 8
\]
Since $\lambda > 0$, $\lambda = 2$, $a = 7/3$ and $b = -3/2$. The osculating circle is
\[
\left( x - \frac{7}{3}\right)^2 + \left( y + \frac{3}{2}\right)^2 = 8
\]