Examples

(1) Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local extreme value of $3$ at $x = -1$ and a local extreme value of $0$ at $x = 1$.

Solution. Since $f(x)$ has local extremes at $-1$ and $1$, $f'(-1) = f'(1) = 0$. Therefore,

$$f'(x) = 3ax^2 + 2bx + c \equiv 3a(x + 1)(x - 1) = 3ax^2 - 3a.$$ 

Therefore, $b = 0$ and $c = -3a$. Then $f(x) = ax^3 - 3ax + d$. Finally, since $f(-1) = 3$ and $f(1) = 0$, we have

$$\begin{align*}
2a + d &= 3 \\
-2a + d &= 0
\end{align*}$$

which gives us $a = 3/4$ and $d = 3/2$. Therefore,

$$f(x) = 3 \cdot \frac{x^3}{4} - \frac{9}{4}x + \frac{3}{2}.$$

(2) Show that the inflection points of the curve $y = x \sin x$ lie on the curve $y^2(x^2 + 4) = 4x^2$.

Proof. Since

$$(x \sin x)'' = (\sin x + x \cos x)'$$

$$= \cos x + \cos x - x \sin x = 2 \cos x - x \sin x$$

the inflection points of $y = x \sin x$ lies on both the curves $y = x \sin x$ and $2 \cos x - x \sin x = 0$. That is, these inflection points satisfy

$$\begin{cases}
  x \sin x = y \\
  2 \cos x - x \sin x = 0
\end{cases} \Rightarrow \begin{cases}
  x \sin x = y \\
  2 \cos x - y = 0
\end{cases} \Rightarrow (2y)^2 + (xy)^2 = (2x)^2$$

Namely, these inflection points lie on $y^2(x^2 + 4) = 4x^2$. 

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