MATHEMATICS 113/114
Midterm Examination, Version 2

Fall 2011

Date: Wednesday, October 26, 2011
Time: 50 minutes

LAST NAME: ___________   FIRST NAME: ______________
(Please, print!)

Please, check your section/instructor!

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<th>Instructor</th>
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<tr>
<td>Math 113, E1</td>
<td>E. Osmanagic</td>
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<tr>
<td>Math 114, D1</td>
<td>X. Chen</td>
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Instructions

1. Books, notes or calculators are not permitted.
2. Show all your work.
3. Make sure your examination paper has 5 questions.
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3. Make sure your examination paper has 5 questions.

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<th>Question</th>
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1. (20 points)

(a) Find the domain of $f$ if $f(x) = \sqrt{x^2 - 4} + \sqrt{8 - x}$

**Solution**

$$D_f = \{ x \mid x^2 - 4 \geq 0 \text{ and } 8 - x \geq 0 \}.$$

Since

$$x^2 - 4 \geq 0 \iff x \in (-\infty, -2] \cup [2, \infty)$$

and

$$8 - x \geq 0 \iff x \in (-\infty, 8],$$

we have

$$D_f = ((-\infty, -2] \cup [2, \infty)) \cap (-\infty, 8] = (-\infty, -2] \cup [2, 8].$$

(b) Find horizontal and vertical asymptotes of $f(x) = \frac{x - 1}{2x + 1}$. Justify your answer.

**Solution:**

$$\lim_{x \to -\infty} \frac{x - 1}{2x + 1} = \lim_{x \to -\infty} \frac{1 - \frac{1}{x}}{2 + \frac{1}{x}} = \frac{1}{2}$$

$$\lim_{x \to \infty} \frac{x - 1}{2x + 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x}}{2 + \frac{1}{x}} = \frac{1}{2}$$

Thus, $y = 1/2$ is a horizontal asymptote as $x \to -\infty$ and as $x \to \infty$.

Note that $x = -\frac{1}{2}$ is the ”candidate” for the vertical asymptote. Since,

$$\lim_{x \to -\frac{1}{2}} \frac{x - 1}{2x + 1} = \infty, \quad \lim_{x \to -\frac{1}{2}} \frac{x - 1}{2x + 1} = -\infty$$

the line $x = -\frac{1}{2}$ is the vertical asymptote.
(c) Evaluate
\[ \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{2 \sin x} \]
Solution:
\[
\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{2 \sin x} = \frac{1}{2} \lim_{x \to 0} \left[ \frac{\sqrt{1 + x} - 1}{\sin x} \cdot \frac{\sqrt{1 + x} + 1}{\sqrt{1 + x} + 1} \right] = \frac{1}{2} \lim_{x \to 0} \frac{1 + x - 1}{\sin x(\sqrt{1 + x} + 1)} = \frac{1}{2} \lim_{x \to 0} \frac{1}{\sin x} \cdot \lim_{x \to 0} \frac{1}{\sqrt{1 + x} + 1} = \frac{1}{4}
\]
(d) Let \( g(x) = e^{3x+1}f(x^3) \). Find \( g'(2) \) if \( f(8) = 1, f'(8) = 2 \).
Solution: By the Chain Rule:
\[
g'(x) = e^{3x+1}3f(x^3) + e^{3x+1}f'(x^3)3x^2 = e^{3x+1}(3f(x^3) + 3x^2 f'(x^3)).
\]
Thus,
\[
g'(2) = e^7(3f(8) + 12f'(8)) = 27e^7.
\]
2. (10 points) Consider the function \( y = f(x) = \frac{3}{\sqrt{x-2}} \).

(a) Use the definition of the derivative to find \( f'(x) \) and state the domain for \( f \) and \( f' \).

NOTE: No marks will be given if the definition is not used.

(b) Find an equation of the normal line to the graph of \( f \) at the point \((a, f(a))\) where \( a = 4 \).

Solution:

(a)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3}{\sqrt{x+h-2}} - \frac{3}{\sqrt{x-2}} = \lim_{h \to 0} \frac{3\sqrt{x-2} - 3\sqrt{x+h-2}}{\sqrt{x+h-2}\sqrt{x-2}} = \lim_{h \to 0} \frac{x - 2 - (x + h - 2)}{h\sqrt{x+h-2} - 2\sqrt{x-2}} = \lim_{h \to 0} \frac{-h}{h\sqrt{x+h-2} - 2\sqrt{x-2}} = \lim_{h \to 0} \frac{-1}{\sqrt{x+h-2} - 2\sqrt{x-2}} = \frac{-3}{2\sqrt{x-2} - 2\sqrt{x-2}} = \frac{-3}{2\sqrt{(x-2)^3}}.
\]

The domain for \( f \) and \( f' \) is the same set, namely \((2, \infty)\).

(b) First note that \( f(4) = \frac{3\sqrt{2}}{2} \). Thus, we need the equation of the normal line at the point \((4, \frac{3\sqrt{2}}{2})\). If \( m_T \) denotes the slope of the tangent line then the slope of the normal line is \(-\frac{1}{m_T}\). On the other hand, \( m_T = f'(4) \). Since by part (a) (or by the Chain Rule, differentiating \( f(x) = 3(x - 2)^{-\frac{1}{2}} \), the derivative \( f'(x) \) of \( f \) is \( f'(x) = -\frac{3}{2}(x - 2)^{-\frac{3}{2}} \) we get \( m_T = f'(4) = -\frac{3}{4\sqrt{2}} \). Hence, the slope of the normal line is \( \frac{4\sqrt{2}}{3} \). Finally, the equation of the equation of the normal line is \( y - \frac{3\sqrt{2}}{2} = \frac{4\sqrt{2}}{3}(x - 4) \).
3. (10 points)

(a) Find all possible values of $a$ and $b$ so that the function

$$f(x) = \begin{cases} 
  x + 2b, & x < 1 \\
  bx - a, & 1 \leq x \leq 2 \\
  4x - 4, & x > 2 
\end{cases}$$

is continuous at every $x$. Justify your answer.

**Solution:** First note that $f$ is continuous on $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$ for all values of $a$ and $b$ as a polynomial degree one. There are two points left, $x = 1$ and $x = 2$.

For $f$ to be continuous at $x = 1$, the following equations must hold:

$$f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x).$$

$$f(1) = b - a$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 2b) = 1 + 2b.$$  

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (bx - a) = b - a$$

Thus

$$b - a = 1 + 2b.$$  

For $f$ to be continuous at $x = 2$, the following equations must hold:

$$f(2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x).$$

$$f(2) = 2b - a$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (bx - a) = 2b - a.$$  

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4x - 4) = 4$$

Thus

$$2b - a = 4.$$  

Finally, $f$ is continuous on the set of real numbers if $b = 1$ and $a = -2$.

(b) Differentiate:

$$f(x) = \cos(\cos(\cos(x^3))).$$

**Solution:** By the Chain Rule,

$$f'(x) = (-\sin(\cos(x^3)))(\cos(x^3))(\cos(\cos(x^3)))(-\sin(x^3))3x^2.$$  

.$$
4. (10 points) Evaluate the limit or explain why the limit does not exist:

(a) \( \lim_{x \to 0} \frac{\sin(2x)}{\tan(3x)} \)

Solution:

\[
\lim_{x \to 0} \frac{\sin(2x)}{\tan(3x)} = \lim_{x \to 0} \frac{\sin(2x)}{\frac{\sin(3x)}{\cos(3x)}} = \lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} \cdot \lim_{x \to 0} \frac{\cos(3x)}{3x} = \lim_{x \to 0} \frac{\sin(2x)}{2} \cdot \frac{2}{3} = \frac{2}{3}
\]

(b) \( \lim_{x \to \infty} (x - \sqrt{x^2 + x}) \)

Solution:

\[
\lim_{x \to \infty} (x - \sqrt{x^2 + x}) = \lim_{x \to \infty} (x - \sqrt{x^2 + x}) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}} = \\
= \lim_{x \to \infty} \frac{-x}{x + x \sqrt{1 + \frac{1}{x}}} = \lim_{x \to \infty} \frac{-x}{x(1 + \sqrt{1 + \frac{1}{x}})} = -\frac{1}{2}
\]
5. (10 points)

A function \( y = y(x) \) is implicitly defined by the equation:

\[
2e^y \cos x - 2 = \sin(xy).
\]

(a) Find \( \frac{dy}{dx} \).

(b) Find the slope of the tangent line to the graph of \( y = y(x) \) at the point \((2\pi, 0)\).

Solution:

(a) Differentiating both sides of the equation with respect to the variable \( x \) taking into account that \( y = y(x) \) we get:

\[
2 \left( e^y \frac{dy}{dx} \cos x + e^y (-\sin x) \right) = \cos(xy) (y + x \frac{dy}{dx}),
\]

or

\[
\frac{dy}{dx} \left( 2e^y \cos x - x \cos(xy) \right) = y \cos(xy) + 2e^y \sin x.
\]

Thus,

\[
\frac{dy}{dx} = \frac{y \cos(xy) + 2e^y \sin x}{2e^y \cos x - x \cos(xy)}, \quad 2e^y \cos x - x \cos(xy) \neq 0.
\]

(b) The slope of the tangent line is:

\[
\frac{dy}{dx} / (2\pi, 0) = \frac{0}{2 - 2\pi} = 0.
\]