

Solution for Midterm Review Problems¹

- (1) Use the definition of derivative to compute the tangent line of the curve $y = 1/x^2$ at the point $(2, 1/4)$.

Solution. Let $f(x) = 1/x^2$. By the definition of derivatives,

$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{h} \\ &= \lim_{h \rightarrow 0} -\frac{(2+h)^2 - 2^2}{2^2(2+h)^2 h} \\ &= -\lim_{h \rightarrow 0} \frac{4h + h^2}{2^2(2+h)^2 h} = -\lim_{h \rightarrow 0} \frac{4+h}{4(2+h)^2} = -\frac{1}{4} \end{aligned}$$

So the tangent line is $y - 1/4 = -(1/4)(x - 2)$.

- (2) (a) Let $f(x) = |\sin(x)|$. Find where $f(x)$ is continuous and where $f(x)$ is differentiable. Justify your answer.

Solution. Since $f(x) = g(h(x))$ with $g(x) = |x|$ and $h(x) = \sin(x)$ and both $g(x)$ and $h(x)$ are continuous functions on $(-\infty, \infty)$, $f(x)$ is continuous everywhere. We have

$$|\sin(x)| = \begin{cases} \sin x & \text{if } 2k\pi \leq x < (2k+1)\pi \\ -\sin x & \text{if } (2k+1)\pi \leq x < (2k+2)\pi \end{cases}$$

for k integers. Obviously, $f(x)$ is differentiable for $x \neq n\pi$; $f(x)$ is not differentiable at $2k\pi$ because

$$(\sin x)'|_{x=2k\pi} = 1 \neq -1 = (-\sin x)'|_{x=2k\pi}$$

and $f(x)$ is not differentiable at $(2k+1)\pi$ because

$$(\sin x)'|_{x=(2k+1)\pi} = -1 \neq 1 = (-\sin x)'|_{x=(2k+1)\pi}$$

In conclusion, $f(x)$ is continuous on $(-\infty, \infty)$ and differentiable on $\{x : x \neq n\pi \text{ for } n \text{ integers}\}$.

¹<http://www.math.ualberta.ca/~xichen/math11406w/p1sol.pdf>

- (b) Let $f(x) = \sin(|x|)$ for $-\pi < x < \pi$. Find where $f(x)$ is continuous and where $f(x)$ is differentiable. Justify your answer.

Solution. Since $f(x) = g(h(x))$ with $g(x) = \sin(x)$ and $h(x) = |x|$ and both $g(x)$ and $h(x)$ are continuous functions on $(-\infty, \infty)$, $f(x)$ is continuous everywhere. We have

$$\sin(|x|) = \begin{cases} \sin x & \text{if } x \geq 0 \\ -\sin x & \text{if } x < 0 \end{cases}$$

Obviously, $f(x)$ is differentiable for $x \neq 0$; $f(x)$ is not differentiable at 0 because

$$(\sin x)'|_{x=0} = 1 \neq -1 = (-\sin x)'|_{x=0}$$

Therefore, $f(x)$ is continuous on $(-\infty, \infty)$ and differentiable on $(-\infty, 0) \cup (0, \infty)$.

- (3) Let

$$f(x) = \begin{cases} -1 - 2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

Sketch the graph of $f(x)$ and find where $f(x)$ is continuous and where $f(x)$ is differentiable.

Solution. $f(x)$ is continuous and differentiable everywhere on $\{x \neq -1, 1\}$. At $x = -1$, $(-1 - 2x)|_{x=-1} = (x^2)|_{x=-1} = 1$ and hence $f(x)$ is continuous; $(-1 - 2x)'|_{x=-1} = (x^2)'|_{x=-1} = -2$ and hence $f(x)$ is differentiable. At $x = 1$, $(x^2)|_{x=1} = (x)|_{x=1} = 1$ and hence $f(x)$ is continuous; $(x^2)'|_{x=1} \neq (x)'|_{x=1}$ and hence $f(x)$ is not differentiable. In conclusion, $f(x)$ is continuous everywhere on $(-\infty, \infty)$ and differentiable on $(-\infty, 1) \cup (1, \infty)$.

- (4) Use Intermediate Value Theorem to show that the equation $\tan(x) = 2x$ has at least one solution in the interval $(0, \pi/2)$.

Proof. Let $f(x) = \tan(x) - 2x$. Since $f(x)$ is continuous on $(0, \pi/2)$, $f(\pi/4) = 1 - \pi/2 < 0$ and $f(\tan^{-1}(\pi)) = \pi - 2\tan^{-1}(\pi) > 0$, there exists a number $c \in (\pi/4, \tan^{-1}(\pi))$ such that $f(c) = 0$ by Intermediate Value Theorem. So $\tan(x) = 2x$ has at least one solution in $(0, \pi/2)$.

- (5) Find the following limits.

$$(a) \lim_{x \rightarrow 1} \frac{x-1}{x^3-1}$$

Solution.

$$\lim_{x \rightarrow 1} \frac{x-1}{x^3-1} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x^2+x+1)} = \lim_{x \rightarrow 1} \frac{1}{x^2+x+1} = \frac{1}{3}$$

$$(b) \lim_{x \rightarrow 2} \frac{x-1}{x^3-1}$$

Solution. Since $(x-1)/(x^3-1)$ is continuous at 2,

$$\lim_{x \rightarrow 2} \frac{x-1}{x^3-1} = \frac{2-1}{2^3-1} = \frac{1}{7}.$$

$$(c) \lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(2x)}$$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(2x)} &= \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{\sin(2x)} \right) \cos(2x) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{\sin(2x)} \right) \lim_{x \rightarrow 0} \cos(2x) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin(3x)/(3x)}{\sin(2x)/(2x)} \right) \left(\frac{3x}{2x} \right) \\ &= \frac{3 \lim_{x \rightarrow 0} \sin(3x)/(3x)}{2 \lim_{x \rightarrow 0} \sin(2x)/(2x)} = \frac{3}{2} \end{aligned}$$

$$(d) \lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{1}{3} - \frac{1}{3+x} \right)$$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{1}{3} - \frac{1}{3+x} \right) &= \lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{(3+x)-3}{3(3+x)} \right) \\ &= \lim_{x \rightarrow 0} \frac{1}{3(3+x)} = \frac{1}{9} \end{aligned}$$

$$(e) \lim_{x \rightarrow 0} x \cos \left(1 + \frac{1}{x} \right)$$

Solution. Since $-1 \leq \cos(1 + 1/x) \leq 1$,

$$-|x| \leq x \cos \left(1 + \frac{1}{x} \right) \leq |x|.$$

By Squeeze theorem,

$$\lim_{x \rightarrow 0} -|x| = \lim_{x \rightarrow 0} |x| = \lim_{x \rightarrow 0} x \cos \left(1 + \frac{1}{x} \right) = 0.$$

(6) Compute

$$\lim_{x \rightarrow 1} \frac{\sqrt[5]{x} - 1}{x - 1}$$

by writing it as the derivative of some function $f(x)$ at some number a .

Solution. Take $f(x) = \sqrt[5]{x}$. Then the limit is $f'(1)$. $f'(x) = (1/5)x^{-4/5}$ and hence $f'(1) = 1/5$, which is the value of the limit.

(7) A table of values for f , g , f' , g' is given as follows.

| x | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|-----|--------|--------|---------|---------|
| 1 | 1 | 2 | 1 | 2 |
| 2 | 2 | 1 | 2 | 1 |

(a) If $h(x) = f(g(x))$, find $h'(1)$.

Solution. By chain rule, $h'(x) = f'(g(x))g'(x)$. So

$$h'(1) = f'(g(1))g'(1) = f'(2) \cdot 2 = 2 \cdot 2 = 4.$$

(b) If $H(x) = g(f(x))$, find $H'(1)$.

Solution. By chain rule, $H'(x) = g'(f(x))f'(x)$. So

$$H'(1) = g'(f(1))f'(1) = g'(1) \cdot 1 = 2 \cdot 1 = 2.$$

(8) Find the first derivatives of the following functions.

(a) $f(x) = \frac{x^2 + x + 1}{\sqrt[3]{x^2}}$

Solution.

$$\begin{aligned} f'(x) &= (x^{4/3} + x^{1/3} + x^{-2/3})' \\ &= \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-5/3} \end{aligned}$$

$$(b) f(x) = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}$$

Solution.

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x} + 1)'(\sqrt{x} - 1) - (\sqrt{x} + 1)(\sqrt{x} - 1)'}{(\sqrt{x} - 1)^2} \\ &= \frac{(1/2)x^{-1/2}(x^{1/2} - 1) - (x^{1/2} + 1)(1/2)x^{-1/2}}{(\sqrt{x} - 1)^2} \\ &= -\frac{x^{-1/2}}{(\sqrt{x} - 1)^2} = -\frac{1}{\sqrt{x}(\sqrt{x} - 1)^2} \end{aligned}$$

$$(c) f(x) = \sin(x^2) \cos(\sqrt{x})$$

Solution.

$$\begin{aligned} f'(x) &= (\sin(x^2))' \cos(\sqrt{x}) + \sin(x^2)(\cos(\sqrt{x}))' \\ &= 2x \cos(x^2) \cos(\sqrt{x}) - \frac{1}{2\sqrt{x}} \sin(x^2) \sin(\sqrt{x}) \end{aligned}$$

$$(d) f(x) = \sqrt{x + \sqrt{x}}$$

Solution.

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x + \sqrt{x}}} (x + \sqrt{x})' \\ &= \frac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right) \end{aligned}$$

- (9) Let $F(x) = (f(x))^3$ and $G(x) = f(x^3)$. If $f(1) = 1$ and $f'(1) = 2$, find $F'(1)$ and $G'(1)$.

Solution. By Chain Rule, $F'(x) = 3f^2(x)f'(x)$ and $G'(x) = f'(x^3)(3x^2)$. So $F'(1) = 3f^2(1)f'(1) = 6$ and $G'(1) = f'(1) \cdot 3 = 6$.

- (10) Suppose that $4 \cos(x) \sin(y) = 1$. Find dy/dx by implicit differentiation.

Solution. Differentiate both sides of the equation with respect to x :

$$\begin{aligned}\frac{d}{dx}(4 \cos(x) \sin(y)) &= 0 \\ \Rightarrow 4 \frac{d}{dx}(\cos(x) \sin(y)) &= 0 \\ \Rightarrow \frac{d}{dx}(\cos(x)) \sin(y) + \cos(x) \frac{d}{dx}(\sin(y)) &= 0 \\ \Rightarrow -\sin(x) \sin(y) + \cos(x) \cos(y) \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= \frac{\sin(x) \sin(y)}{\cos(x) \cos(y)} = \tan(x) \tan(y)\end{aligned}$$