Math 114 Midterm Solution[1]

(1) Let

$$f(x) = \left| \frac{x^3}{x-1} \right|$$

Find where $f(x)$ is continuous and where $f(x)$ is differentiable. Justify your answer.

Solution. The domain of $f(x)$ is $(-\infty, 1) \cup (1, \infty)$. Since $f = g \circ h$ where $g(x) = |x|$ and $h(x) = x^3/(x-1)$ are continuous functions on their domains, $f(x)$ is continuous on $(-\infty, 1) \cup (1, \infty)$.

Since $x^3/(x-1) < 0$ when $0 < x < 1$ and $x^3/(x-1) \ge 0$ when $x \le 0$ or $x > 1$,

$$\left| \frac{x^3}{x-1} \right| = \begin{cases} \frac{x^3}{x-1} & \text{if } x \le 0 \text{ or } x > 1 \\ -\frac{x^3}{x-1} & \text{if } 0 < x < 1 \end{cases}$$

and hence $f(x)$ is differentiable for $x \ne 0, 1$. We have

$$\left(\frac{x^3}{x-1} \right)' = \frac{(x^3)'(x-1) - x^3(x-1)'}{(x-1)^2} = \frac{x^2(2x-3)}{(x-1)^2}$$

and hence

$$\left. \left(\frac{x^3}{x-1} \right)' \right|_{x=0} = \left. \left(-\frac{x^3}{x-1} \right)' \right|_{x=0} = 0$$

Therefore, $f(x)$ is differentiable at 0.

In conclusion, $f(x)$ is continuous and differentiable everywhere on its domain $(-\infty, 1) \cup (1, \infty)$.

(2) Use Intermediate Value Theorem to show that the equation $x^5 + x + 2006 = 0$ has at least one solution.

Proof. Let $f(x) = x^5 + x + 2006$. Since $f(x)$ is a polynomial, it is continuous everywhere. And since $f(0) = 2006 > 0$ and $f(-10) < 0$, there is a number c in $(-10, 0)$ such that $f(c) = 0$ by IVT. Therefore, $f(x) = 0$ has at least one solution.

(3) Suppose that $\sin x + \sin y = 1$. Find dy/dx using implicit differentiation.

Solution. Differentiate both sides with respect to x

$$\frac{d}{dx}(\sin x + \sin y) = \frac{d}{dx}(1) \Rightarrow \cos(x) + \cos(y)\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{\cos(x)}{\cos(y)}$$

(4) Find the following limits if they exist.

[1] http://www.math.ualberta.ca/~xichen/math11406w/midsol.pdf

(a) $\displaystyle\lim_{x\to 2^-} \frac{x^3-8}{x^2-5x+6}$

Solution.

$$\begin{aligned}\lim_{x\to 2^-} \frac{x^3-8}{x^2-5x+6} &= \lim_{x\to 2^-} \frac{(x-2)(x^2+2x+4)}{(x-2)(x-3)} \\ &= \lim_{x\to 2^-} \frac{x^2+2x+4}{x-3} = -12\end{aligned}$$

(b) $\displaystyle\lim_{x\to 0} \frac{x-\sin(3x)}{\tan(x)}$

Solution.

$$\begin{aligned}\lim_{x\to 0} \frac{x-\sin(3x)}{\tan(x)} &= \lim_{x\to 0} \frac{x-\sin(3x)}{\sin(x)/\cos(x)} = \left(\lim_{x\to 0} \frac{x-\sin(3x)}{\sin(x)}\right)\left(\lim_{x\to 0} \cos x\right) \\ &= \lim_{x\to 0} \frac{x-\sin(3x)}{\sin(x)} = \lim_{x\to 0} \frac{x}{\sin x} - \lim_{x\to 0} \frac{\sin(3x)}{\sin(x)} \\ &= \frac{1}{\displaystyle\lim_{x\to 0} \frac{\sin x}{x}} - \left(\lim_{x\to 0} \frac{\sin(3x)/(3x)}{\sin(x)/x} \cdot \frac{3x}{x}\right) = 1-3 = -2\end{aligned}$$

(c) $\displaystyle\lim_{x\to 0^+} x\sin\left(\frac{1+x}{x}\right)$

Solution. Since

$$-1 \le \sin\left(\frac{1+x}{x}\right) \le 1,$$

$$-x \le x\sin\left(\frac{1+x}{x}\right) \le x$$

for $x > 0$. And since

$$\lim_{x\to 0^+} (-x) = \lim_{x\to 0^+} x = 0,$$

$$\lim_{x\to 0^+} x\sin\left(\frac{1+x}{x}\right) = 0$$

by squeeze theorem.

(d) $\displaystyle\lim_{x\to (-1)^+} \frac{x+1}{|x^2-1|}$

Solution.

$$\begin{aligned}\lim_{x\to (-1)^+} \frac{x+1}{|x^2-1|} &= \left(\lim_{x\to (-1)^+} \frac{x+1}{|x+1|}\right)\left(\lim_{x\to (-1)^+} \frac{1}{|x-1|}\right) \\ &= \left(\lim_{x\to (-1)^+} \frac{x+1}{x+1}\right)\left(\frac{1}{2}\right) = \frac{1}{2}\end{aligned}$$

(5) Find the derivatives of the following functions.

(a) $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

Solution.

$$\begin{aligned}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)' &= \frac{(\sqrt{x}-1)'(\sqrt{x}+1)-(\sqrt{x}-1)(\sqrt{x}+1)'}{(\sqrt{x}+1)^2} \\ &= \frac{(\sqrt{x}+1)-(\sqrt{x}-1)}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\end{aligned}$$

(b) $f(x) = \sqrt{x\sqrt{x}}$

Solution.

$$\begin{aligned}\left(\sqrt{x\sqrt{x}}\right)' &= \left(\sqrt{x^{3/2}}\right)' \\ &= \left(x^{3/4}\right)' = \frac{3}{4}x^{-1/4}\end{aligned}$$

(c) $f(x) = \cos(x \sin x)$

Solution.

$$\begin{aligned}(\cos(x\sin x))' &= -\sin(x\sin x)(x\sin x)' \\ &= -\sin(x\sin x)((x)'\sin x + x(\sin x)') \\ &= -\sin(x\sin x)(\sin x + x\cos x)\end{aligned}$$

(d) $f(x) = x\cos\left(\dfrac{1}{x}\right)$

Solution.

$$\begin{aligned}\left(x\cos\left(\frac{1}{x}\right)\right)' &= (x)'\cos\left(\frac{1}{x}\right) + x\left(\cos\left(\frac{1}{x}\right)\right)' \\ &= \cos\left(\frac{1}{x}\right) - x\left(\frac{1}{x}\right)'\sin\left(\frac{1}{x}\right) \\ &= \cos\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)\end{aligned}$$