Math 114 Final Review[1]

(1) Compute the following limits if they exist.

(a) $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

Solution.

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \lim_{x\to 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{4}$$

(b) $\lim_{x\to -\infty} \frac{\sin(2x)}{x}$

Solution. Since $-1 \le \sin(2x) \le 1$,

$$-\frac{1}{x} \ge \frac{\sin(2x)}{x} \ge \frac{1}{x}$$

when $x < 0$. And since

$$\lim_{x\to -\infty} -\frac{1}{x} = \lim_{x\to -\infty} \frac{1}{x} = 0,$$

$$\lim_{x\to -\infty} \frac{\sin(2x)}{x} = 0$$

by squeeze theorem.

(c) $\lim_{x\to \infty} \frac{1-3x^2+2x^3}{1-x^3}$

Solution.

$$\lim_{x\to \infty} \frac{1-3x^2+2x^3}{1-x^3} = \lim_{x\to \infty} \frac{\frac{1}{x^3} - \frac{3}{x} + 2}{\frac{1}{x^3} - 1}$$
$$= \frac{\lim_{x\to\infty} \frac{1}{x^3} - \lim_{x\to\infty} \frac{3}{x} + \lim_{x\to\infty} 2}{\lim_{x\to\infty} \frac{1}{x^3} - \lim_{x\to\infty} 1} = -2$$

(d) $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

Solution.

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0} \frac{(\cos\theta - 1)(\cos\theta + 1)}{\sin\theta(\cos\theta + 1)}$$
$$= -\lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta} = 0$$

(2) Find the local and absolute maxima and minima of the function $f(x) = \sin x + \cos x$ on $[-\pi, \pi]$.

Solution. Solve $f'(x) = 0 \Leftrightarrow \cos x - \sin x = 0 \Leftrightarrow \tan x = 1 \Leftrightarrow x = k\pi + \pi/4$. Hence $f(x)$ has two critical numbers $-3\pi/4$ and $\pi/4$. Since $f(-\pi) = f(\pi) = -1$, $f(-3\pi/4) = -\sqrt{2}$ and $f(\pi/4) = \sqrt{2}$, $f(x)$ achieves the absolute maximum $\sqrt{2}$ when $x = \pi/4$ and the absolute minimum $-\sqrt{2}$ when $x = -3\pi/4$; it has a local maximum at $\pi/4$ and a local minimum at $-3\pi/4$.

[1]http://www.math.ualberta.ca/~xichen/math11406w/fpsol.pdf

(3) (a) Show that the equation $x^3 + x + 3 = 0$ has exactly one real root.
Proof. Let $f(x) = x^3 + x + 3$. It is continuous and differentiable everywhere on $(-\infty, \infty)$. And since $f(0) = 3 > 0$ and $f(-2) = -7 < 0$, there is a number c in $(-2, 0)$ such that $f(c) = 0$ by Intermediate Value Theorem. Therefore, $f(x) = 0$ has at least one root. Suppose that $f(x) = 0$ has two distinct roots a and b, i.e., $f(a) = f(b) = 0$ and $a < b$. By Mean Value Theorem, there is a number c in (a, b) such that

$$f'(c) = \frac{f(a) - f(b)}{a - b} = 0$$

On the other hand, $f'(x) = 3x^2 + 1 > 0$ for all x. Contradiction. Therefore, $f(x) = 0$ has at most one real root. In conclusion, $f(x) = 0$ has exactly one real root.

(b) Use Newton's method with initial approximation $x_1 = -1$ to find x_3, the third appromixation to the root of the equation $x^3 + x + 3 = 0$.
Solution. We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 3}{3x_n^2 + 1} = \frac{2x_n^3 - 3}{3x_n^2 + 1}$$

So

$$x_2 = \frac{2x_1^3 - 3}{3x_1^2 + 1} = -\frac{5}{4} \text{ and } x_3 = \frac{2x_2^3 - 3}{3x_2^2 + 1} = -\frac{221}{182}$$

(4) Sketch the graph of the function

$$f(x) = \frac{x^2 + 1}{x + 1}$$

You must follow the steps A-H as in Sec. 4.5: (A) Domain (B) Intercepts (C) Symmetry (D) Asmptotes (E) Intervals of Increases and Decreases (F) Local maximum and minimum (G) Concavity and points of inflection (H) Sketch the curve.
Solution.

(a) Domain: $\{x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$

(b) x-intercepts: $f(x) = 0$ has no solution; so no x-intercepts. y-intercepts: $(0, f(0)) = (0, 1)$.

(c) Since

$$f(-x) = \frac{x^2 + 1}{-x + 1}$$

$f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So $f(x)$ is neither even nor odd and it is not periodic.

(d) Since

$$\lim_{x \to (-1)^-} \frac{x^2 + 1}{x + 1} = -\infty \text{ and } \lim_{x \to (-1)^+} \frac{x^2 + 1}{x + 1} = \infty$$

$y = f(x)$ has a vertical asymptote $x = -1$. To find the slant asymptotes, we compute

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2+1}{x(x+1)}$$
$$= \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{1+\frac{1}{x}} = 1$$

$$b = \lim_{x\to\infty}(f(x) - kx) = \lim_{x\to\infty}\left(\frac{x^2+1}{x+1} - x\right)$$
$$= \lim_{x\to\infty} \frac{1-x}{x+1} = \lim_{x\to\infty} \frac{\frac{1}{x}-1}{1+\frac{1}{x}} = -1$$

and

$$k = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{x^2+1}{x(x+1)}$$
$$= \lim_{x\to-\infty} \frac{1+\frac{1}{x^2}}{1+\frac{1}{x}} = 1$$

$$b = \lim_{x\to-\infty}(f(x) - kx) = \lim_{x\to-\infty}\left(\frac{x^2+1}{x+1} - x\right)$$
$$= \lim_{x\to-\infty} \frac{1-x}{x+1} = \lim_{x\to-\infty} \frac{\frac{1}{x}-1}{1+\frac{1}{x}} = -1$$

Therefore, the only slant asymptote is $y = x - 1$.

(e) Since

$$f'(x) = \frac{(x^2+1)'(x+1) - (x^2+1)(x+1)'}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$$
$$= \frac{(x-(-1-\sqrt{2}))(x-(-1+\sqrt{2}))}{(x+1)^2}$$

$f(x)$ is increasing on $(-\infty, -1-\sqrt{2})$ and $(-1+\sqrt{2}, \infty)$ and it is decreasing on $(-1-\sqrt{2}, -1)$ and $(-1, -1+\sqrt{2})$.

(f) Since $f'(x)$ changes from $+$ to $-$ at $-1-\sqrt{2}$, $f(x)$ has a local maximum at $-1-\sqrt{2}$; since $f'(x)$ changes from $-$ to $+$ at $-1+\sqrt{2}$, $f(x)$ has a local minimum at $-1+\sqrt{2}$.

(g) Since

$$f''(x) = \left(\frac{x^2+1}{x+1}\right)'' = \left(x - 1 + \frac{2}{x+1}\right)'' = \frac{4}{(x+1)^3}$$

$f(x)$ is concave upward on $(-1, \infty)$ and downward on $(-\infty, -1)$. It has no point of inflection.

(5) Compute the following integrals.

(a) $\displaystyle\int_1^4 \frac{x^2+x+1}{\sqrt{x}}dx$

Solution.

$$\int_1^4 \frac{x^2+x+1}{\sqrt{x}}dx = \int_1^4 (x^{3/2}+x^{1/2}+x^{-1/2})dx$$
$$= \left(\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+2x^{1/2}\right)\Big|_1^4 = \frac{286}{15}$$

(b) $\displaystyle\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

Solution. Substitute $t = 1+2x$:

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} \frac{1}{2}t^{-2/3}dt$$
$$= \frac{3}{2}t^{1/3}\Big|_1^{27} = 3$$

(c) $\displaystyle\int_1^2 x\sqrt{x-1}dx$

Solution. Substitute $t = \sqrt{x-1}$:

$$\int_1^2 x\sqrt{x-1}dx = \int_0^1 (t^2+1)td(t^2+1)$$
$$= \int_0^1 2t^2(t^2+1)dt = 2\int_0^1 (t^4+t^2)dt$$
$$= \left(\frac{2}{5}t^5+\frac{2}{3}t^3\right)\Big|_0^1 = \frac{16}{15}$$

(6) Find an equation of the line through the point $(3,5)$ that cuts off the least area from the first quadrant.

Solution. Let k be the slope of the line. It is obvious that $k<0$. The equation of the line is

$$y-5 = k(x-3)$$

So the xy-intercepts of the line are $(0, 5-3k)$ and $(3-5/k, 0)$. Therefore, the area of the triangle cut out by the line is

$$\frac{1}{2}(5-3k)(3-\frac{5}{k}) = \frac{1}{2}(30-9k-\frac{25}{k}) = \frac{1}{2}f(k)$$

It suffices to minimize $f(k)$ on $(-\infty, 0)$:

$$f'(k) = 0 \Leftrightarrow -9+\frac{25}{k^2} = 0 \Leftrightarrow k = \pm\frac{5}{3}$$

So $f(k)$ has a critical point at $k=-5/3$. And since

$$\lim_{x\to-\infty} f(k) = \lim_{x\to 0^-} f(k) = \infty$$

$f(k)$ achieves the minimum when $k=-5/3$. Therefore, the line that cuts off the least area is

$$y-5 = -\frac{5}{3}(x-3)$$

(7) At noon, Ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm?
Solution. We set up the coordinate system with A at the origin and B at $(100, 0)$ at noon. After t hours, A and B are at $(0, -35t)$ and $(100, 25t)$, respectively. Therefore, the distance between A and B is

$$f(t) = \sqrt{(0 - 100)^2 + (-35t - 25t)^2} = \sqrt{100^2 + 60^2 t^2} = 20\sqrt{25 + 9t^2}$$

The derivative of $f(t)$ is

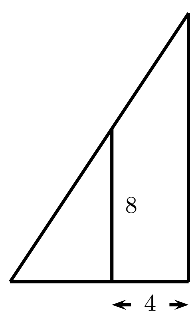
$$f'(t) = \frac{180t}{\sqrt{25 + 9t^2}}$$

So the distance between the ships is changing at a rate of

$$f'(4) = \frac{720}{\sqrt{169}} = \frac{720}{13} \text{ km/h}$$

at 4:00 pm.

(8) A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



Solution. Let θ be the angle between the ladder and the ground. Then the length of the ladder is given by

$$f(\theta) = \frac{8}{\sin\theta} + \frac{4}{\cos\theta}$$

Minimize $f(\theta)$ on $(0, \pi/2)$:

$$f'(\theta) = 0 \Rightarrow -\frac{8\cos\theta}{\sin^2\theta} + \frac{4\sin\theta}{\cos^2\theta} = 0$$
$$\Rightarrow 8\cos^3\theta = 4\sin^3\theta \Rightarrow \tan^3\theta = 2$$
$$\Rightarrow \tan\theta = \sqrt[3]{2}$$

Since

$$\lim_{\theta\to 0^+} f(\theta) = \lim_{\theta\to(\pi/2)^-} f(\theta) = \infty$$

$f(\theta)$ takes the minimum when $\tan\theta = \sqrt[3]{2}$. When $\tan\theta = \sqrt[3]{2}$,

$$f(\theta) = \frac{8\sqrt{1 + \sqrt[3]{4}}}{\sqrt[3]{2}} + 4\sqrt{1 + \sqrt[3]{4}} = 4(1 + \sqrt[3]{4})^{3/2}$$

which is the length of the shortest ladder.

(9) Find the tangent line to the curve $x^2 + xy + 2y^2 = 4$ at the point $(1, 1)$.
Solution. Use implicit differentiation:

$$\frac{d}{dx}(x^2 + xy + 2y^2) = \frac{d}{dx}(4) \Rightarrow 2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dx}{dy} = -\frac{2x + y}{x + 4y} \Rightarrow \left.\frac{dx}{dy}\right|_{(1,1)} = -\frac{3}{5}$$

Therefore, the tangent line at $(1, 1)$ is

$$y - 1 = -\frac{3}{5}(x - 1)$$

(10) Let $F(x) = \sqrt{f(x)}$ and $G(x) = f(\sqrt{x})$. If $f(1) = 1$, $f'(1) = 2$ and $f''(1) = 3$, find $F''(1)$ and $G''(1)$.
Solution.

$$F''(x) = (F'(x))' = \left(\frac{1}{2}(f(x))^{-1/2} f'(x)\right)'$$
$$= \frac{1}{2}\left(-\frac{1}{2}(f(x))^{-3/2}(f'(x))^2 + (f(x))^{-1/2} f''(x)\right)$$

So $F''(1) = 1/2$.

$$G''(x) = (G'(x))' = \left(\frac{1}{2}f'(\sqrt{x})x^{-1/2}\right)'$$
$$= \frac{1}{2}\left(\frac{1}{2}f''(\sqrt{x})x^{-1} - \frac{1}{2}f'(\sqrt{x})x^{-3/2}\right)$$

So $G''(1) = 1/4$.

(11) Consider the integral

$$\int_{-2}^{3}(1 - 2x)dx$$

(a) Write the above integral as a limit of Riemann sums.
Solution. Let $f(x) = 1 - 2x$.

$$\int_{-2}^{3}(1 - 2x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_0 + k\Delta x)\Delta x$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{5}{n}f(-2 + \frac{5k}{n})$$
$$= \lim_{n\to\infty}\frac{5}{n}\sum_{k=1}^{n}\left(5 - \frac{10k}{n}\right) = \lim_{n\to\infty}\frac{25}{n}\sum_{k=1}^{n}\left(1 - \frac{2k}{n}\right)$$

(b) Compute the limit you obtained in part (a). You may need the formula:

$$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$$

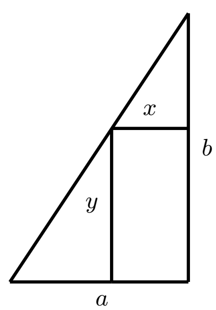
Solution.

$$\lim_{n\to\infty}\frac{25}{n}\sum_{k=1}^{n}\left(1-\frac{2k}{n}\right) = \lim_{n\to\infty}\frac{25}{n}\left(\sum_{k=1}^{n}1-\frac{2}{n}\sum_{k=1}^{n}\right)$$
$$= \lim_{n\to\infty}\frac{25}{n}(n-(n+1)) = -\lim_{n\to\infty}\frac{25}{n} = 0$$

(c) Verify your answer by computing the integral using Fundamental Theorem of Calculus.
Solution.

$$\int_{-2}^{3}(1-2x)dx = (x-x^2)\Big|_{-2}^{3} = (-6)-(-6) = 0$$

(12) Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths a and b if two sides of the rectangle lie along the legs.



Solution. Let x and y be the length and width of the rectangle (see the figure above). Then

$$\frac{x}{a} = \frac{b-y}{b} \Rightarrow x = \frac{a}{b}(b-y)$$

Therefore, the area of the rectangle is

$$xy = \frac{a}{b}(b-y)y = f(y)$$

Maximize $f(y)$ on $[0, b]$:

$$f'(y) = 0 \Rightarrow \frac{a}{b}(b-2y) = 0 \Rightarrow y = b/2$$

So $f(y)$ has a critical points at $y = b/2$. Since $f(0) = f(b) = 0$ and $f(b/2) = ab/4$, $f(y)$ takes the maximum at $y = b/2$. The largest area of the rectangle is $ab/4$.