Solution for Midterm Review Problems[1]

(1) Use the definition of derivative to compute the tangent line of the curve $y = 1/x$ at the point $(2, 1/2)$.

Let $f(x) = 1/x$. By the definition of derivatives,

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{1}{2+h} - \dfrac{1}{2}}{h} \\ &= \lim_{h\to 0} \frac{-\dfrac{h}{2(2+h)}}{h} \\ &= -\lim_{h\to 0} \frac{1}{2(2+h)} = -\frac{1}{4} \end{aligned}$$

So the tangent line is $y - 1/2 = -(1/4)(x-2)$.

(2) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

First of all, $f(x)$ is obviously differentiable everywhere outside of $x = 2$. To make $f(x)$ differentiable at $x = 2$, we have to make sure

(a) $f(x)$ is continuous at $x = 2$, which implies that the values of x^2 and $mx + b$ agree at $x = 2$;

(b) the derivatives of x^2 and $mx + b$ agree at $x = 2$.

So we have $(x^2)|_{x=2} = (mx+b)|_{x=2}$ and $(x^2)'|_{x=2} = (mx+b)'|_{x=2}$, i.e., $4 = 2m + b$ and $4 = m$, which yields $m = 4$ and $b = -4$.

(3) Let

$$f(x) = \begin{cases} -1 - 2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}.$$

Sketch the graph of $f(x)$ and find where $f(x)$ is continuous and where $f(x)$ is differentiable.

[1] http://www.math.ualberta.ca/~xichen/math11403f/p1sol.pdf

$f(x)$ is continuous and differentiable everywhere on $\{x \neq -1, 1\}$. At $x = -1$, $(-1 - 2x)|_{x=-1} = (x^2)|_{x=-1} = 1$ and hence $f(x)$ is continuous; $(-1 - 2x)'|_{x=-1} = (x^2)'|_{x=-1} = -2$ and hence $f(x)$ is diffentiable. At $x = 1$, $(x^2)|_{x=1} = (x)|_{x=1} = 1$ and hence $f(x)$ is continuous; $(x^2)'|_{x=1} \neq (x)'|_{x=1}$ and hence $f(x)$ is not differentiable. In conclusion, $f(x)$ is continuous everywhere on $(-\infty, \infty)$ and differentiable on $(-\infty, 1) \cup (1, \infty)$.

(4) Use Intermediate Value Theorem to show that the equation $\tan(x) = 2x$ has at least one solution in the interval $(0, \pi/2)$.

Let $f(x) = \tan(x) - 2x$. Since $f(x)$ is continuous on $(0, \pi/2)$, $f(\pi/4) = 1 - \pi/2 < 0$ and $f(\tan^{-1}(\pi)) = \pi - 2\tan^{-1}(\pi) > 0$, there exists a number $c \in (\pi/4, \tan^{-1}(\pi))$ such that $f(c) = 0$ by Intermediate Value Theorem. So $\tan(x) = 2x$ has at least one solution in $(0, \pi/2)$.

(5) Find the following limits.

(a) $\displaystyle\lim_{x \to 1} \frac{x-1}{x^3-1}$

$$\lim_{x \to 1} \frac{x-1}{x^3-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{1}{x^2+x+1} = \frac{1}{3}$$

(b) $\displaystyle\lim_{x \to 2} \frac{x-1}{x^3-1}$

Since $(x-1)/(x^3-1)$ is continuous at 2,

$$\lim_{x \to 2} \frac{x-1}{x^3-1} = \frac{2-1}{2^3-1} = \frac{1}{7}.$$

(c) $\displaystyle\lim_{x \to 0} \frac{\sin(3x)}{\tan(2x)}$

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(3x)}{\tan(2x)} &= \lim_{x \to 0} \left(\frac{\sin(3x)}{\sin(2x)}\right) \cos(2x) \\
&= \lim_{x \to 0} \left(\frac{\sin(3x)}{\sin(2x)}\right) \lim_{x \to 0} \cos(2x) \\
&= \lim_{x \to 0} \left(\frac{\sin(3x)/(3x)}{\sin(2x)/(2x)}\right) \left(\frac{3x}{2x}\right) \\
&= \frac{3 \lim_{x \to 0} \sin(3x)/(3x)}{2 \lim_{x \to 0} \sin(2x)/(2x)} = \frac{3}{2}
\end{aligned}$$

(d) $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{3} - \frac{1}{3+x}\right)$

$$\begin{aligned}\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{3} - \frac{1}{3+x}\right) &= \lim_{x\to 0} \frac{1}{x}\left(\frac{(3+x)-3}{3(3+x)}\right)\\ &= \lim_{x\to 0} \frac{1}{3(3+x)} = \frac{1}{9}\end{aligned}$$

(e) $\lim_{x\to 0} x\cos\left(1 + \frac{1}{x}\right)$

Since $-1 \le \cos(1+1/x) \le 1$,

$$-|x| \le x\cos\left(1+\frac{1}{x}\right) \le |x|.$$

By Squeeze theorem,

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = \lim_{x\to 0} x\cos\left(1+\frac{1}{x}\right) = 0.$$

(6) Compute

$$\lim_{x\to 1} \frac{\sqrt[5]{x}-1}{x-1}$$

by writing it as the derivative of some function $f(x)$ at some number a.

Take $f(x) = \sqrt[5]{x}$. Then the limit is $f'(1)$. $f'(x) = (1/5)x^{-4/5}$ and hence $f'(1) = 1/5$, which is the value of the limit.

(7) (20 points) A table of values for f, g, f', g' is given as follows.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	1	2	1	2
2	2	1	2	1

(a) If $h(x) = f(g(x))$, find $h'(1)$.

By chain rule, $h'(x) = f'(g(x))g'(x)$. So

$$h'(1) = f'(g(1))g'(1) = f'(2)\cdot 2 = 2\cdot 2 = 4.$$

(b) If $H(x) = g(f(x))$, find $H'(1)$.

By chain rule, $H'(x) = g'(f(x))f'(x)$. So

$$H'(1) = g'(f(1))f'(1) = g'(1)\cdot 1 = 2\cdot 1 = 2.$$

(8) Find the first derivatives of the following functions.

(a) $f(x) = \dfrac{x^2 + x + 1}{\sqrt[3]{x^2}}$

$$\begin{aligned} f'(x) &= (x^{4/3} + x^{1/3} + x^{-2/3})' \\ &= \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-5/3} \end{aligned}$$

(b) $f(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x}+1)'(\sqrt{x}-1) - (\sqrt{x}+1)(\sqrt{x}-1)'}{(\sqrt{x}-1)^2} \\ &= \frac{(1/2)x^{-1/2}(x^{1/2}-1) - (x^{1/2}+1)(1/2)x^{-1/2}}{(\sqrt{x}-1)^2} \\ &= -\frac{x^{-1/2}}{(\sqrt{x}-1)^2} = -\frac{1}{\sqrt{x}(\sqrt{x}-1)^2} \end{aligned}$$

(c) $f(x) = \sin(x^2)\cos(\sqrt{x})$

$$\begin{aligned} f'(x) &= (\sin(x^2))'\cos(\sqrt{x}) + \sin(x^2)(\cos(\sqrt{x}))' \\ &= 2x\cos(x^2)\cos(\sqrt{x}) - \frac{1}{2\sqrt{x}}\sin(x^2)\sin(\sqrt{x}) \end{aligned}$$

(d) $f(x) = \sqrt{x + \sqrt{x}}$

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x+\sqrt{x}}}(x + \sqrt{x})' \\ &= \frac{1}{2\sqrt{x+\sqrt{x}}}(1 + \frac{1}{2\sqrt{x}}) \end{aligned}$$

(9) Let $F(x) = (f(x))^3$ and $G(x) = f(x^3)$. If $f(1) = 1$ and $f'(1) = 2$, find $F'(1)$ and $G'(1)$.

By Chain Rule, $F'(x) = 3f^2(x)f'(x)$ and $G'(x) = f'(x^3)(3x^2)$. So $F'(1) = 3f^2(1)f'(1) = 6$ and $G'(1) = f'(1) \cdot 3 = 6$.