Solution for Midterm[1]

(1) $f(x)$ is continuous outside of -1 and 1. At -1, $(x^2)|_{x=-1} \neq (-2x)|_{x=-1}$ so $f(x)$ is neither continuous nor differentiable at -1. At 1, $(x^2)|_{x=1} = (2x-1)|_{x=1}$ and $(x^2)'|_{x=1} = (2x-1)'|_{x=1}$ so $f(x)$ is continuous and differentiable at 1. So $f(x)$ is continuous and differentiable on $\{x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$.

(2) Let $f(x) = x^3 - x + 1$. Since $f(x)$ is continuous on $(-\infty, \infty)$, $f(1) = 1 > 0$ and $f(-2) = -5 < 0$, by Intermediate Value Theorem, there exists a c in $(-2, 1)$ such that $f(c) = 0$.

(3) Take $f(x) = \sqrt[4]{x}$ and $a = 16$. Then the limit is $f'(16)$. Since $f'(x) = (1/4)x^{-3/4}$, $f'(16) = 1/32$.

(4) By Chain Rule, $h'(x) = f'(g(x))g'(x)$ and $H'(x) = g'(f(x))f'(x)$. So $h'(1) = f'(g(1))g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$ and $H'(1) = g'(f(1))f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

(5) (a)

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2}(x+2) = 4\end{aligned}$$

(b) Since $(x^2 - 4)/(x - 2)$ is continuous at 1, the limit is the value of the function at 1, which is $(1^2 - 4)/(1 - 2) = 3$.

(c)

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} &= \lim_{x\to 0} \frac{\sin(3x)/(3x)}{\sin(2x)/(2x)} \left(\frac{3x}{2x}\right) \\ &= \left(\frac{3}{2}\right) \frac{\lim_{x\to 0}\sin(3x)/(3x)}{\lim_{x\to 0}\sin(2x)/(2x)} = \frac{3}{2}\end{aligned}$$

(d) Since $-1 \leq \sin(1/x) \leq 1$, $-|x| \leq x\sin(1/x) \leq |x|$. So by Squeeze theorem,

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0.$$

(6) (a) $(t^2 + 2t - 1)' = 2t + 2$.

(b)

$$\begin{aligned}\left(\frac{x+1}{x-1}\right)' &= \frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2} \\ &= -\frac{2}{(x-1)^2}\end{aligned}$$

[1] http://www.math.ualberta.ca/~xichen/math11403f/mid1sol.pdf

(c)

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\begin{aligned}
(\sin(\cos(\sqrt{x})))' &= \cos(\cos(\sqrt{x})) \cdot \big(\cos(\sqrt{x})\big)' \\
&= -\cos(\cos(\sqrt{x}))\sin(\sqrt{x})(\sqrt{x})' \\
&= -\frac{1}{2\sqrt{x}}\cos(\cos(\sqrt{x}))\sin(\sqrt{x})
\end{aligned}
$$

(d) $(\sqrt{x\sqrt{x}})' = \big((x \cdot x^{1/2})^{1/2}\big)' (x^{3/4})' = (3/4)x^{-1/4}.$