# Solution for Final Review Problems[1]

(1) Compute the following limits.

(a) $\lim_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2-1})$

$$\begin{aligned}&\lim_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2-1})\\ &=\lim_{x\to\infty}\frac{(\sqrt{x^2+1}-\sqrt{x^2-1})(\sqrt{x^2+1}+\sqrt{x^2-1})}{\sqrt{x^2+1}+\sqrt{x^2-1}}\\ &=\lim_{x\to\infty}\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}=0\end{aligned}$$

(b) $\lim_{x\to 1}\frac{x^3-x}{x^3-1}$

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-x}{x^3-1}&=\lim_{x\to 1}\frac{(x-1)(x^2+x)}{(x-1)(x^2+x+1)}\\ &=\lim_{x\to 1}\frac{x^2+x}{x^2+x+1}=\frac{2}{3}\end{aligned}$$

(c) $\lim_{x\to-\infty}\frac{6x^2+5x}{(1-x)(2x-3)}$

$$\lim_{x\to-\infty}\frac{6x^2+5x}{(1-x)(2x-3)}=\lim_{x\to-\infty}\frac{6+5/x}{(1/x-1)(2-3/x)}=-3$$

(d) $\lim_{x\to 0}\frac{\sqrt{1-x}-\sqrt{1+x}}{x}$

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1-x}-\sqrt{1+x}}{x}&=\lim_{x\to 0}\frac{(\sqrt{1-x}-\sqrt{1+x})(\sqrt{1-x}+\sqrt{1+x})}{x(\sqrt{1-x}+\sqrt{1+x})}\\ &=\lim_{x\to 0}\frac{-2x}{x(\sqrt{1-x}+\sqrt{1+x})}=-1\end{aligned}$$

(e) $\lim_{t\to 2}\frac{t^{-1}-2^{-1}}{t-2}$

$$\begin{aligned}\lim_{t\to 2}\frac{t^{-1}-2^{-1}}{t-2}&=\lim_{t\to 2}\frac{\frac{2-t}{2t}}{t-2}\\ &=\lim_{t\to 2}-\frac{1}{2t}=-\frac{1}{4}\end{aligned}$$

[1] http://www.math.ualberta.ca/~xichen/math11403f/fp1sol.pdf

(f) $\lim_{x\to 0} \frac{\sin(4x)}{\tan(3x)}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(4x)}{\tan(3x)} &= \lim_{x\to 0} \left(\frac{\sin(4x)}{\sin(3x)}\right)\cos(3x)\\ &= \left(\lim_{x\to 0} \frac{\sin(4x)}{\sin(3x)}\right)\lim_{x\to 0}\cos(3x)\\ &= \lim_{x\to 0}\left(\frac{\sin(4x)/(4x)}{\sin(3x)/(3x)}\right)\left(\frac{4x}{3x}\right) = \frac{4}{3}\end{aligned}$$

(2) Find the derivative of each of the following functions.

(a) $f(x) = x\tan(x) + \cos(x^2)$

$$f'(x) = (x\tan(x))' + (\cos(x^2))' = \tan(x) + x\sec^2(x) - 2x\sin(x^2)$$

(b) $f(x) = \frac{x^3+1}{x^3-1}$

$$\begin{aligned}f'(x) &= \frac{(x^3+1)'(x^3-1) - (x^3+1)(x^3-1)'}{(x^3-1)^2}\\ &= \frac{3x^2(x^3-1) - 3x^2(x^3+1)}{(x^3-1)^2} = -\frac{6x^2}{(x^3-1)^2}\end{aligned}$$

(c) $f(t) = \frac{6}{\sqrt[3]{t^5}}$

$$f'(t) = (6t^{-5/3})' = -10t^{-8/3}$$

(d) $f(x) = \sqrt{\cos(\sin(x))}$

$$f'(x) = -\frac{\sin(\sin(x))\cos(x)}{2\sqrt{\cos(\sin(x))}}$$

(3) Find local and absolute maxima and minima of the function $f(x) = x^3 - 3x$ on the interval $[-2, 2]$.

Take the derivative of $f(x)$: $f'(x) = 3x^2 - 3$. Solve $f'(x) = 0$ and we obtain two critical points $x = 1$ and $x = -1$. Compare $f(-2)$, $f(-1)$, $f(1)$ and $f(2)$ and we see that $f(x)$ takes the absolute maximum 2 when $x = 2$ or $-1$ and $f(x)$ takes the absolute minimum $-2$ when $x = -2$ or 1. And $f(x)$ has a local maximum at $x = -1$ and a local minimum at $x = 1$.

(4) A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is his shadow on the building decreasing when he is 4 m from the building?

Let $x$ be the distance between the spotlight and the man and $y$ be the length of his shadow on the building (both in meters). Then $2/y = x/12$ and hence $xy = 24$. Take derivative on both sides of $xy = 24$ with respect to $t$:

$$x\left(\frac{dy}{dt}\right) + \left(\frac{dx}{dt}\right)y = 0 \Rightarrow \frac{dy}{dt} = -\frac{y}{x}\left(\frac{dx}{dt}\right)$$

Plug in $dx/dt = 1.6$, $x = 8$ and $y = 3$ and we obtain $dy/dt = -0.6$. So his shadown is decreasing at a rate of 0.6 m/s.

(5) Use Intermediate Value Theorem and Mean Value Theorem to show that the equation $x^5 + x^3 + x + 1 = 0$ has exactly one solution.

Let $f(x) = x^5 + x^3 + x + 1$. Since $f(0) = 1$ and $f(-1) = -2$, $f(x) = 0$ has a solution in $(0, 1)$ by Intermdediate Value Theorem. Next, we will show this solution is unique.

Suppose that $f(x) = 0$ has two solutions $x = a$ and $x = b$. So $f(a) = f(b) = 0$. By Mean Value Theorem (or Rolle's Theorem), there exists $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

On the other hand, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all $x$. This is a contradiction. Hence $f(x) = 0$ has only one solution.

(6) The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm$^2$, find the dimension of the poster with the smallest area.

Let $x$ and $y$ be the length and width of the poster (measured in cm). Then the printed area is $(x - 8)(y - 12) = 384$. So

$$y = 12 + \frac{384}{x - 8}$$

The area of the poster is

$$S = xy = 12x + \frac{384x}{x - 8}$$

Solve $S'(x) = 0$:

$$12 - 384\left(\frac{8}{(x - 8)^2}\right) = 0 \Rightarrow (x - 8)^2 = 256 \Rightarrow x = 24$$

Since $\lim_{x\to 8^+} S(x) = \lim_{x\to\infty} S(x) = \infty$, $S(x)$ takes the minimum at $x = 24$ on $(8, \infty)$. So the smallest poster has dimension $24 \times 36$.

(7) A cylindrical can is made out of materials costing 6 dollar/$m^2$ for the top and bottom and 10 dollar/$m^2$ for the sides. Assume that its volume is fixed at 10 $m^3$. Find the dimension that minimizes the cost.

Let $r$ be the radius of the base of the can and $h$ be its height (in meters). Then the total cost is

$$6(2\pi r^2) + 10(2\pi rh) = 12\pi r^2 + 20\pi rh$$

with volume $\pi r^2 h = 10$. Substitute $h = 10/(\pi r^2)$ and the cost function becomes

$$f(r) = 12\pi r^2 + \frac{200}{r}$$

Solve $f'(r) = 0$:

$$24\pi r - \frac{200}{r^2} = 0 \Rightarrow r = \sqrt[3]{\frac{25}{3\pi}}$$

Since $\lim_{r\to 0^+} f(r) = \lim_{r\to\infty} = \infty$, $f(r)$ takes the minimum at $r = \sqrt[3]{\frac{25}{3\pi}}$. So the cheapest can has base radius $\sqrt[3]{\frac{25}{3\pi}}$ m and height $\sqrt[3]{\frac{72}{5\pi}}$ m.

(8) Two cars start moving from the same point. One travels south at 60 mph and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

Let $f(t)$ be the distance between two cars after $t$ hours. Then $f(t) = \sqrt{(60t)^2 + (25t)^2} = 65t$. So $f'(t) = 65$ and the distance between the cars is increasing at a rate of 65 mph.

(9) Sketch the graphs of each of the following functions. You must follow the steps A-H as in Sec. 4.5: (A) Domain (B) Intercepts (C) Symmetry (D) Asymptotes (E) Intervals of Increases and Decreases (F) Local maximum and minimum (G) Concavity and points of inflection (H) Sketch the curve.

(a) $f(x) = 8x^2 - x^4$

The domain of $f(x)$ is $(-\infty, \infty)$. Solve $f(x) = 0$ and we obtain the $x$-intercepts: $(-2\sqrt{2}, 0)$, $(0, 0)$ and $(2\sqrt{2}, 0)$. And the $y$-intercept is $(0, f(0)) = (0, 0)$.

Since $f(-x) = 8(-x)^2 - (-x)^4 = 8x^2 - x^4 = f(x)$, $f(x)$ is even. It is not odd and not periodic.

The graph $y = f(x)$ does not have any vertical asymptotes. Since

$$\lim_{x\to\infty} \frac{8x^2 - x^4}{x} = \lim_{x\to\infty} (8x - x^3) = -\infty$$

and

$$\lim_{x\to-\infty}\frac{8x^2-x^4}{x}=\lim_{x\to\infty}(8x-x^3)=\infty$$

it does not have any slant and horizontal asymptotes.
Take the first derivative of $f(x)$: $f'(x)=16x-4x^3=-4x(x-2)(x+2)$. Therefore, $f(x)$ is increasing for $x$ in $(-\infty,-2)\cup(0,2)$ and decreasing for $x$ in $(-2,0)\cup(2,\infty)$. Since $f'(x)$ changes from positive to negative at $-2$, $f(x)$ has a local maximum at $-2$. Since $f'(x)$ changes from negative to positive at 0, $f(x)$ has a local minimum at 0. By symmetry, $f(x)$ has a local maximum at 2.
Take the second derivative of $f(x)$, $f''(x)=16-12x^2=-12(x-2/\sqrt{3})(x+2/\sqrt{3})$. Hence $f(x)$ is concave upward for $x$ in $(-2/\sqrt{3},2/\sqrt{3})$; $f(x)$ is concave downward for $x$ in $(-\infty,-2/\sqrt{3})\cup(2/\sqrt{3},\infty)$. And $2/\sqrt{3}$ and $-2/\sqrt{3}$ are points of inflection.

(b) $f(x)=\dfrac{1+x}{1-x}$

The domain of $f(x)$ is $(-\infty,1)\cup(1,\infty)$. The $x$-intercept is $(-1,0)$ and the $y$-intercept is $(0,1)$. It is not even, odd and periodic.
The graph of $y=f(x)$ has a vertical asymptote $x=1$. Since

$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{f(x)}{x}=0$$

and

$$\lim_{x\to\infty}f(x)=\lim_{x\to-\infty}f(x)=-1$$

it has a horizontal asymptote $y=-1$.
Take the first derivative of $f(x)$: $f'(x)=2/(1-x)^2$. So $f(x)$ is always increasing and it does not have any local maxima/minima.
Take the second derivative of $f(x)$: $f''(x)=4/(1-x)^3$. So $f(x)$ is concave upward for $x$ in $(-\infty,1)$ and concave downward for $x$ in $(1,\infty)$. It does not have any points of inflection.

(10) Evaluate the following integrals.

(a) $\int_0^1(x^2-x)dx$

$$\int_0^1(x^2-x)dx=\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Bigg|_0^1=-\frac{1}{6}$$

(b) $\int_0^{\pi/2}(\sin(x) - \cos(x))dx$

$$\int_0^{\pi/2}(\sin(x) - \cos(x))dx = (-\cos(x) - \sin(x))|_0^{\pi/2} = 0$$

(c) $\int \sqrt[2002]{1+3x}dx$
Substitute $t = 1 + 3x$, i.e., $x = (t-1)/3$

$$\int \sqrt[2002]{1+3x}dx = \int t^{1/2002} d\left(\frac{t-1}{3}\right)$$
$$= \frac{2002}{6009}t^{2003/2002} + C = \frac{2002}{6009}(1+3x)^{2003/2002} + C$$

(d) $\displaystyle\int \frac{x^2}{(x^3+1)^2}dx$
Substitute $t = x^3 + 1$ ($x^2dx = \frac{1}{3}d(x^3+1)$)

$$\int \frac{x^2}{(x^3+1)^2}dx = \frac{1}{3}\int \frac{1}{(x^3+1)^2}d(x^3+1) = \frac{1}{3}\int \frac{1}{t^2}dt$$
$$= -\frac{1}{3t} + C = -\frac{1}{3(x^3+1)} + C$$

(11) Express the following integral as a limit of Riemann sums. Do not evaluate the limit.

(a) $\displaystyle\int_2^6 \frac{x}{1+x^5}dx$
Divide $[2,6]$ into $n$ intervals of length $\Delta x = 4/n$:

$$\int_2^6 f(x)dx = \int_2^6 \frac{x}{1+x^5}dx$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x f(x_i) = \lim_{n\to\infty}\Delta x\sum_{i=1}^{n} f(2+i\Delta x)$$
$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} f\left(\frac{2n+4i}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4n^3(2n+4i)}{n^5+(2n+4i)^5}$$

(b) $\int_0^{2\pi} x^2\sin(x)dx$

Divide $[0, 2\pi]$ into $n$ intervals of length $\Delta x = 2\pi/n$:

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} x^2 \sin(x)dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(0 + i\Delta x)$$

$$= \lim_{n\to\infty} \frac{2\pi}{n} \sum_{i=1}^{n} f\left(\frac{2\pi i}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8\pi^3 i^2}{n^3} \sin\left(\frac{2\pi i}{n}\right)$$

(12) The table gives the values of a function obtained from experiment. Use them to estimate $\int_0^6 f(x)dx$ using midpoint rule with $n = 3$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 2 | 4 | 6 | 2 | 1 |

The midpoint approximation of the integral is

$$2(f(1) + f(3) + f(5)) = 2(3 + 4 + 2) = 18$$

(13) Starting with $x_1 = -1$ use Newton's method to find $x_3$, the third appromixation to the root of the equation $x^3 + x + 3 = 0$.

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 3}{3x_n^2 + 1} = \frac{2x_n^3 - 3}{3x_n^2 + 1}$$

So

$$x_2 = \frac{2x_1^3 - 3}{3x_1^2 + 1} = \frac{2(-1)^3 - 3}{3(-1)^2 + 1} - \frac{5}{4}$$

and

$$x_3 = \frac{2x_2^3 - 3}{3x_2^2 + 1} = \frac{2(-5/4)^3 - 3}{3(-5/4)^2 + 1} = -\frac{221}{107}$$

(14) For what values of $x$ does the graph of $f(x) = x + \sin(x)$ have a horizontal tangent?

Solve $f'(x) = 1 + \cos(x) = 0$ and we obtain $x = 2k\pi + \pi$. So the graph of $f(x)$ has a horizontal tangent when $x = (2k + 1)\pi$ for all integers $k$.

(15) Let $F(x) = \sqrt[3]{f(x)}$ and $G(x) = f(\sqrt[3]{x})$. If $f(1) = 2$ and $f'(1) = 3$, find $F'(1)$ and $G'(1)$.

Since $F'(x) = \frac{1}{3}[f(x)]^{-2/3} f'(x)$, $F'(1) = 2^{-2/3} = \sqrt[3]{2}/2$.

Since $G'(x) = f'(\sqrt[3]{x})x^{-2/3}/3$, $G'(1) = 1$.

(16) The displacement of a particle is given by $s(t) = A\cos(Bt + C)$ with $A, B, C$ constants. Find the velocity and acceleration of the particle at time $t$.

The velocity is $f'(t) = -AB\sin(Bt+C)$ and the acceleration is $f''(t) = -AB^2\cos(Bt+C)$.