Midterm Solution[1]

(1) Since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} c - 3x = c - 3, \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - c = 1 - c$$

we must have $c - 3 = 1 - c$ for $f(x)$ to be continuous. So $c = 2$.

(2) Let x be the distance between the batter and the home base, y be the distance between the batter and the first base and z be the distance between the batter and the third base (all in ft). Then

$$(90 - x)^2 + 90^2 = y^2.$$

Differentiate both sides with respect to t:

$$-2(90 - x)\frac{dx}{dt} = 2y\frac{dy}{dt}$$

where $dx/dt = 24$. When $x = 45$, $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$. So

$$\frac{dy}{dt} = -\frac{90 - x}{y}\frac{dx}{dt} = -\frac{45}{45\sqrt{5}}(24) = -\frac{24\sqrt{5}}{5}$$

when $x = 45$. Similarly,

$$x^2 + 90^2 = z^2$$

and differentiate both sides with respect to t:

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}.$$

When $x = 45$, $z = \sqrt{45^2 + 90^2} = 45\sqrt{5}$. So $dz/dt = 24\sqrt{5}/5$.

(3) Compute $f'(4)$ by the definition of derivatives:

$$\begin{aligned} f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} \\ &= \lim_{x\to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4} \end{aligned}$$

So the tangent line is $y - 2 = (x - 4)/4$.

(4) (a) $\displaystyle\lim_{x\to -3} \frac{x^2 - x - 12}{x + 3} = \lim_{x\to -3} \frac{(x + 3)(x - 4)}{x + 3} = \lim_{x\to -3}(x - 4) = -7$

(b) $\displaystyle\lim_{x\to \pi/4} \frac{\sin 2x}{\tan x} = \frac{\sin(\pi/2)}{\tan(\pi/4)} = 1$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{3}{2}\left(\frac{\sin(3x)}{3x}\right) / \left(\frac{\sin(2x)}{2x}\right) = \frac{3}{2}$

[1] http://www.math.ualberta.ca/~xichen/math11402f/mid1sol.pdf

(d)

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\lim_{x\to 1}\left(\frac{x+1}{(x-1)(x+1)}-\frac{2}{x^2-1}\right)$$
$$=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$$

(e) $\displaystyle\lim_{x\to -4^-}\frac{|x+4|}{x+4}=\lim_{x\to 4^-}\frac{x+4}{x+4}=1$

(5) (a) $f'(t)=(t^{2/3}+2t^{3/2})'=t^{-1/3}+2t^{1/2}$

(b) $\displaystyle f'(x)=\left(\frac{x^5}{x^3-1}\right)'=\frac{(x^5)'(x^3-1)-x^5(x^3-1)'}{(x^3-1)^2}=\frac{2x^7-5x^4}{(x^3-1)^2}$

(c) $\displaystyle f'(x)=\cos(\cos(\sqrt{x}))(\cos(\sqrt{x}))'=-\frac{\cos(\cos(\sqrt{x}))\sin(\sqrt{x})}{2\sqrt{x}}$

(d) $\displaystyle f'(x)=(\sqrt{x^{3/2}})'=(x^{3/4})'=\frac{3}{4}x^{-1/4}$

(6) Since $F'(x)=(\sqrt{f(x)})'=(1/2)(f(x))^{-1/2}f'(x)$ and $G'(x)=(f(\sqrt{x}))'=(1/2)f'(\sqrt{x})x^{-1/2}$, $F'(1)=(1/2)(f(1))^{-1/2}f'(1)=3\sqrt{2}/4$ and $G'(1)=(1/2)f'(1)(1)^{-1/2}=3/2$.

(7) Differentiate both sides with repect to x:

$$\frac{d}{dx}(2\sin x\cos y)=\frac{d}{dx}(1)\Rightarrow 2\cos x\cos y-2\sin x\sin y\frac{dy}{dx}=0$$
$$\Rightarrow \sin x\sin y\frac{dy}{dx}=\cos x\cos y\Rightarrow\frac{dy}{dx}=\frac{\cos x\cos y}{\sin x\sin y}$$

When $x=y=\pi/4$,

$$\frac{dy}{dx}=\frac{\cos(\pi/4)\cos(\pi/4)}{\sin(\pi/4)\sin(\pi/4)}=1.$$

So the tangent line is

$$y-\frac{\pi}{4}=x-\frac{\pi}{4},\ \text{i.e., } y=x$$