Midterm Solution

(1) Since
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} c - 3x = c - 3, \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x - c = 1 - c
\]
we must have \(c - 3 = 1 - c\) for \(f(x)\) to be continuous. So \(c = 2\).

(2) Let \(x\) be the distance between the batter and the home base, \(y\) be the distance between the batter and the first base and \(z\) be the distance between the batter and the third base (all in ft). Then
\[
(90 - x)^2 + 90^2 = y^2.
\]
Differentiate both sides with respect to \(t\):
\[
-2(90 - x) \frac{dx}{dt} = 2y \frac{dy}{dt}
\]
where \(dx/dt = 24\). When \(x = 45\), \(y = \sqrt{45^2 + 90^2} = 45\sqrt{5}\). So
\[
\frac{dy}{dt} = -\frac{90 - x}{y} \frac{dx}{dt} = -\frac{45}{45\sqrt{5}}(24) = -\frac{24\sqrt{5}}{5}
\]
when \(x = 45\). Similarly,
\[
x^2 + 90^2 = z^2
\]
and differentiate both sides with respect to \(t\):
\[
2x \frac{dx}{dt} = 2z \frac{dz}{dt}.
\]
When \(x = 45\), \(z = \sqrt{45^2 + 90^2} = 45\sqrt{5}\). So \(dz/dt = 24\sqrt{5}/5\).

(3) Compute \(f'(4)\) by the definition of derivatives:
\[
f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}
\]
\[
= \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}
\]
So the tangent line is \(y - 2 = (x - 4)/4\).

(4) (a) \(\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x - 4)}{x + 3} = \lim_{x \to -3} (x - 4) = -7\)

(b) \(\lim_{x \to \pi/4} \frac{\sin(2x)}{\tan(x)} = \frac{\sin(\pi/2)}{\tan(\pi/4)} = 1\)

(c) \(\lim_{x \to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x \to 0} \frac{3}{2} \left(\frac{\sin(3x)}{3x}\right) / \left(\frac{\sin(2x)}{2x}\right) = \frac{3}{2}\)

\(^1\)http://www.math.ualberta.ca/~xichen/math11402f/mid1sol.pdf
(d) \[
\lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) = \lim_{x \to 1} \left( \frac{x + 1}{(x - 1)(x + 1)} - \frac{2}{x^2 - 1} \right) = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}
\]

(e) \[
\lim_{x \to 4^-} \left| \frac{x + 4}{x + 4} \right| = \lim_{x \to 4^-} \frac{x + 4}{x + 4} = 1
\]

(5) (a) \[
f'(t) = \left( t^{2/3} + 2t^{3/2} \right)' = t^{-1/3} + 2t^{1/2}
\]
(b) \[
f'(x) = \left( \frac{x^5}{x^3 - 1} \right)' = \frac{(x^5)'(x^3 - 1) - x^5(x^3 - 1)'}{(x^3 - 1)^2} = \frac{2x^7 - 5x^4}{(x^3 - 1)^2}
\]
(c) \[
f'(x) = \cos(\cos(\sqrt{x}))(\cos(\sqrt{x}))' = -\frac{\cos(\cos(\sqrt{x})) \sin(\sqrt{x})}{2\sqrt{x}}
\]
(d) \[
f'(x) = \left( \sqrt{x^{3/2}} \right)' = (x^{3/4})' = \frac{3}{4}x^{-1/4}
\]

(6) Since \( F'(x) = (\sqrt{f(x)})' = (1/2)(f(x))^{-1/2} f'(x) \) and \( G'(x) = (f(\sqrt{x}))' = (1/2) f'(\sqrt{x}) x^{-1/2} \), \( F'(1) = (1/2)(f(1))^{-1/2} f'(1) = 3\sqrt{2}/4 \) and \( G'(1) = (1/2) f'(1)(1)^{-1/2} = 3/2 \).

(7) Differentiate both sides with respect to \( x \):
\[
\frac{d}{dx}(2 \sin x \cos y) = \frac{d}{dx}(1) \Rightarrow 2 \cos x \cos y - 2 \sin x \sin y \frac{dy}{dx} = 0
\]
\[
\Rightarrow \sin x \sin y \frac{dy}{dx} = \cos x \cos y \Rightarrow \frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \sin y}
\]

When \( x = y = \pi/4 \),
\[
\frac{dy}{dx} = \frac{\cos(\pi/4) \cos(\pi/4)}{\sin(\pi/4) \sin(\pi/4)} = 1.
\]

So the tangent line is
\[
y - \frac{\pi}{4} = x - \frac{\pi}{4}, \text{ i.e., } y = x
\]