Outline

1. How Derivatives Affect the Shape of a Graph
2. Curve Sketching
Find where \( f(x) = x^3 - 3x \) is increasing and where it is decreasing.

Solution. Its derivative is \( f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1) \). Since \( f'(x) > 0 \) on \((-\infty, -1)\) and \((1, \infty)\), \( f(x) \) is increasing on \((-\infty, -1]\) and \([1, \infty)\). Since \( f'(x) < 0 \) on \((-1, 1)\), \( f(x) \) is decreasing on \([-1, 1]\).

Find where \( f(x) = \sin x \) is increasing and where it is decreasing.

Solution. Its derivative is \( f'(x) = \cos x \). Since \( f'(x) > 0 \) on \((2n\pi - \pi/2, 2n\pi + \pi/2)\), \( f(x) \) is increasing on \([2n\pi - \pi/2, 2n\pi + \pi/2]\) for \( n \) integer. Since \( f'(x) < 0 \) on \((2n\pi + \pi/2, 2n\pi + 3\pi/2)\), \( f(x) \) is decreasing on \([2n\pi + \pi/2, 2n\pi + 3\pi/2]\) for \( n \) integer.
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Intervals of Increase and Decrease

1. Find where $f(x) = x^3 - 3x$ is increasing and where it is decreasing.
   Solution. Its derivative is $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$.
   Since $f'(x) > 0$ on $(-\infty, -1)$ and $(1, \infty)$, $f(x)$ is increasing on $(-\infty, -1]$ and $[1, \infty)$. Since $f'(x) < 0$ on $(-1, 1)$, $f(x)$ is decreasing on $[-1, 1]$.

2. Find where $f(x) = \sin x$ is increasing and where it is decreasing.
   Solution. Its derivative is $f'(x) = \cos x$. Since $f'(x) > 0$ on $(2n\pi - \pi/2, 2n\pi + \pi/2)$, $f(x)$ is increasing on $[2n\pi - \pi/2, 2n\pi + \pi/2]$ for $n$ integer. Since $f'(x) < 0$ on $(2n\pi + \pi/2, 2n\pi + 3\pi/2)$, $f(x)$ is decreasing on $[2n\pi + \pi/2, 2n\pi + 3\pi/2]$ for $n$ integer.
How Derivatives Affect the Shape of a Graph
Curve Sketching

**f’(x) and Local Extremes**

**First Derivative Test**

If \( f'(x_0) = 0 \) and \( f'(x) \) changes from positive to negative at \( x_0 \) (\( f'(x) > 0 \) on \((x_0 - r, x_0)\) and \( f'(x) < 0 \) on \((x_0, x_0 + r)\)), then \( f(x) \) has a local maximum at \( x_0 \). If \( f'(x_0) = 0 \) and \( f'(x) \) changes from negative to positive at \( x_0 \) (\( f'(x) < 0 \) on \((x_0 - r, x_0)\) and \( f'(x) > 0 \) on \((x_0, x_0 + r)\)), then \( f(x) \) has a local minimum at \( x_0 \). If \( f'(x) \) does not change sign at \( x_0 \) (\( f'(x) > 0 \) on \((x_0 - r, x_0)\) \( \cup \) \((x_0, x_0 + r)\) or \( f'(x) < 0 \) on \((x_0 - r, x_0)\) \( \cup \) \((x_0, x_0 + r)\)).

- Find the local maxima and minima of \( f(x) = x^3 - 3x \).
- Find the local maxima and minima of \( f(x) = \sin x \).
First Derivative Test

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- Find the local maxima and minima of \( f(x) = \sin x \).
**$f''(x)$ and Concavity**

We say $f(x)$ concave upward (CU) on an interval $I$ if the curve $y = f(x)$ is always above its tangent lines:

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

for all $x_0, x$ in $I$. We say $f(x)$ concave downward (CD) on an interval $I$ if the curve $y = f(x)$ is always below its tangent lines:

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0)$$

for all $x_0, x$ in $I$.

**Criterion for CU and CD**

On $(a, b)$, $f''(x) > 0 \Rightarrow f(x)$ is CU; $f''(x) < 0 \Rightarrow f(x)$ is CD. If $f(x)$ changes from CU to CD or from CD to CU at $x_0$, $P = (x_0, f(x_0))$ is an inflection point of $y = f(x)$. 
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Examples

- Find the intervals of increase/decrease and CU/CD, local extremes and inflection points of \( f(x) = x^3 - 6x^2 + 12x \).

  Solution. Since \( f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 > 0 \) for all \( x \neq 2 \), \( f(x) \) is increasing on \( (-\infty, 2] \) and \([2, \infty) \). So it is increasing on \( (-\infty, \infty) \) and it has no local extremes. Since \( f''(x) = 6x - 12 \), \( f(x) \) is CU on \((2, \infty) \) and CD on \((-\infty, 2) \) and \( P = (2, 8) \) is the inflection point of \( y = f(x) \).

- Find the intervals of increase/decrease and CU/CD, local extremes and inflection points of \( f(x) = \sin x \).

  Solution. Since \( f'(x) = \cos x \), \( f(x) \) in increasing on \((2n\pi - \pi/2, 2n\pi + \pi/2) \) and decreasing on \((2n\pi + \pi/2, 2n\pi + 3\pi/2) \). It has local maxima at \( 2n\pi + \pi/2 \) and local minima at \( 2n\pi - \pi/2 \). Since \( f''(x) = -\sin x \), \( f(x) \) is CU on \((2n\pi - \pi, 2n\pi) \) and CD on \((2n\pi, 2n\pi + \pi) \) and \((m\pi, 0) \) are the inflection points.
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- Find the intervals of increase/decrease and CU/CD, local extremes and inflection points of \( f(x) = x^3 - 6x^2 + 12x \).
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2nd Derivative Test

Suppose that $f'(x_0) = 0$. Then

1. If $f''(x_0) < 0$, $f(x)$ has a local max at $x_0$.
2. If $f''(x_0) > 0$, $f(x)$ has a local min at $x_0$.
3. If $f''(x_0) = 0$, we cannot determine whether $f(x)$ has a local max/min at $x_0$ by this test (we need higher derivatives $f^{(n)}(x_0)$ and Taylor series).

Find the local max/min of $f(x) = x^3 - 3x$.
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**f''(x)** and Local Extremes

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\( f''(x) \) and Local Extremes

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# Summary

<table>
<thead>
<tr>
<th>Test</th>
<th>Purpose</th>
<th>How it works</th>
</tr>
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</table>
| Increasing/Decreasing | Test Monotonicity    | $f'(x) > 0 \Rightarrow f(x) \uparrow$
                   |                      | $f'(x) < 0 \Rightarrow f(x) \downarrow$ |
| **CU/CD**             | Test Concavity       | $f''(x) > 0 \Rightarrow f(x) \text{ CU}$
                   |                      | $f''(x) < 0 \Rightarrow f(x) \text{ CD}$ |
| 1st Derivative        | Determine local max/min | $f'(x)$ goes $-\rightarrow + \Rightarrow$ local min |
                   |                      | $f'(x)$ goes $+\rightarrow - \Rightarrow$ local max |
| 2nd Derivative        | Determine local max/min | $f''(x_0) > 0 \Rightarrow$ local min |
                   |                      | $f''(x_0) < 0 \Rightarrow$ local max |
To sketch the curve $y = f(x)$, we follow these steps:

A. Determine the domain.
B. Determine $xy$-intercepts.
C. Symmetry: determine whether $f(x)$ is odd, even and/or periodic.
D. Asymptotes: find horizontal and vertical asymptotes.
E. Find intervals of increase and decrease.
F. Find local extremes.
G. Concavity and inflection points.
H. Sketch the curve.
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Examples

- Sketch the graph of $f(x) = x^3 - 3x$.
- Sketch the graph of $f(x) = x + \frac{1}{x}$.
- Sketch the graph of $f(x) = \sqrt{x^2 + x - x}$.

Solution.

$$f'(x) = \frac{2x + 1}{2\sqrt{x^2 + x}} - 1 \text{ and } f''(x) = -\frac{1}{4}(x^2 + x)^{-3/2}$$
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f'(x) = \frac{2x + 1}{2\sqrt{x^2 + x}} - 1 \text{ and } f'''(x) = -\frac{1}{4}(x^2 + x)^{-3/2}
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