Outline

1. Derivatives of Inverse Function
2. Logarithmic Differentiation
3. Related Rates
Find \((f^{-1}(x))'\) using Implicit Differentiation

Since \(y = f^{-1}(x)\), \(x = f(y)\). Differentiate both sides with respect to \(x\):

\[
\frac{d}{dx} x = \frac{d}{dx} f(y) \Rightarrow 1 = f'(y) \frac{dy}{dx}
\]

\[\text{Chain Rule}\]

\[
\Rightarrow (f^{-1}(x))' = \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}
\]
Examples of Derivatives of Inverse Functions

- Find \((f^{-1}(x))'\) for \(f(x) = x^3\).

  Solution. Of course, \(f^{-1}(x) = \sqrt[3]{x}\). Let \(y = f^{-1}(x) = \sqrt[3]{x}\). Then

  \[ y = f^{-1}(x) \Leftrightarrow x = f(y) \Rightarrow \frac{dy}{dx} x = \frac{d}{dx} f(y) \Rightarrow 1 = f'(y) \frac{dy}{dx} \]

  Since \(f'(y) = 3y^2\), \((f^{-1}(x))' = \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2} = \frac{1}{3^{\frac{3}{2}}x^2}\).

- Find \((f^{-1})'(2)\) for \(f(x) = x^3 + x\).

  Solution. Let \(y = f^{-1}(x)\). Then

  \[(f^{-1}(x))' = \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2 + 1}\]. It remains to figure out \(f^{-1}(2)\). If \(y = f^{-1}(2)\), then \(2 = f(y) = y^3 + y\). Solve the equation \(y^3 + y - 2 = 0\) and we obtain \(y = 1 = f^{-1}(2)\). So \((f^{-1})'(2) = \frac{1}{4}\).
Examples of Derivatives of Inverse Functions

Find \((f^{-1}(x))'\) for \(f(x) = x^3\).
Solution. Of course, \(f^{-1}(x) = \sqrt[3]{x}\). Let \(y = f^{-1}(x) = \sqrt[3]{x}\). Then

\[
y = f^{-1}(x) \iff x = f(y) \implies \frac{d}{dx} x = \frac{d}{dx} f(y) \implies 1 = f'(y) \frac{dy}{dx}
\]

Since \(f'(y) = 3y^2\), \((f^{-1}(x))' = \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2} = \frac{1}{3\sqrt[3]{x^2}}\).

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Solution. Let \(y = f^{-1}(x)\). Then

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  Solution. Of course, \(f^{-1}(x) = 3\sqrt{x}\). Let \(y = f^{-1}(x) = 3\sqrt{x}\). Then
  \[
y = f^{-1}(x) \iff x = f(y) \implies \frac{dy}{dx}x = \frac{d}{dx}f(y) \implies 1 = f'(y) \frac{dy}{dx}
  \]
  Since \(f'(y) = 3y^2\),
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Derivative of \( \ln x \)

Since \( \ln x \) and \( e^x \) are inverse functions to each other,

\[
y = \ln x \iff x = e^y
\]

Differentiate both sides with respect to \( x \):

\[
\frac{d}{dx} x = \frac{d}{dx} e^y \implies 1 = e^y \frac{dy}{dx} \implies \left( \ln x \right)' = \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.
\]

\[
(\log_a x)' = \left( \frac{\ln x}{\ln a} \right)' = \frac{1}{x \ln a}.
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Derivative of In $x$

Since ln $x$ and $e^x$ are inverse functions to each other,

$$y = \ln x \iff x = e^y$$

Differentiate both sides with respect to $x$:

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$$(\log_a x)' = \left( \frac{\ln x}{\ln a} \right)' = \frac{1}{x \ln a}.$$
Examples

- **Find** \( (\ln(\cos(\log_2(x))))') \).
  
  Solution. By chain rule,
  \[
  (\ln(\cos(\log_2(x))))' = -\frac{\sin(\log_2(x))}{(x \ln 2) \cos(\log_2(x))}
  \]

- Use \( (\ln x)' \) to derive Power Rule.
  
  Solution. Use \( x = e^{\ln x} \):
  \[
  (x^r)' = ((e^{\ln x})^r)' = (e^{r \ln x})' = e^{r \ln x}(r \ln x)' = x^r(rx^{-1}) = rx^{r-1}.
  \]

- **Find** \( (x^x)' \).
  
  Solution. It is WRONG that \( (x^x)' = xx^{x-1} \) or \( (x^x)' = x^x \ln x \).
  \[
  (x^x)' = ((e^{\ln x})^x)' = (e^{x \ln x})' = e^{x \ln x}(x \ln x)' = x^x(\ln x + 1).
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- **Use** \((\ln(x))'\) to derive **Power Rule.**
  **Solution.** Use \(x = e^{\ln x}\):
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  \]
Examples

- **Find** \((\ln(\cos(\log_2(x))))\)′.
  Solution. By chain rule,
  \[
  (\ln(\cos(\log_2(x))))′ = -\frac{\sin(\log_2(x))}{(x \ln 2) \cos(\log_2(x))}
  \]

- **Use** \((\ln x)′\) **to derive** Power Rule.
  Solution. Use \(x = e^{\ln x}\):
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Logarithmic Differentiation

Given $y = f(x)$, it is sometime convenient to take ln on both sides and then apply implicit differentiation:

$$y = f(x) \Rightarrow \ln y = \ln f(x) \Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} (\ln f(x))$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} (\ln f(x)) \Rightarrow \frac{dy}{dx} = f(x) \frac{d}{dx} (\ln f(x)).$$

This technique is called *Logarithmic Differentiation*. Usually, we apply it together with the basic identities of exponential and logarithmic functions such as

$$\ln(ab) = \ln a + \ln b, \quad \ln \frac{a}{b} = \ln a - \ln b$$

and

$$\ln a^b = b \ln a.$$
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$$\ln(ab) = \ln a + \ln b, \quad \ln \frac{a}{b} = \ln a - \ln b \quad \text{and} \quad \ln a^b = b \ln a.$$
Examples of Logarithmic Differentiation

Find \((x^x) '\).  
Solution. Let \(y = x^x\). Then

\[
\ln y = \ln(x^x) \Rightarrow \ln y = x \ln x \Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)
\]

\[
\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x + 1 \Rightarrow \frac{dy}{dx} = y(\ln x + 1) \Rightarrow (x^x)' = x^x(\ln x + 1).
\]

Find \(\left((1 + \cos x)^{\sin x}\right)'\).  
Solution. Let \(y = (1 + \cos x)^{\sin x}\). Then

\[
\ln y = \ln((1 + \cos x)^{\sin x}) \Rightarrow \ln y = (\sin x) \ln(1 + \cos x)
\]

\[
\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} ((\sin x) \ln(1 + \cos x))
\]

\[
\Rightarrow \frac{dy}{dx} = (1 + \cos x)^{\sin x} \left(\cos x \ln(1 + \cos x) - \frac{\sin x)^2}{1 + \cos x}\right).
\]
Examples of Logarithmic Differentiation

- **Find** $(x^x)'$.
  **Solution.** Let $y = x^x$. Then

  \[ \ln y = \ln(x^x) \Rightarrow \ln y = x \ln x \Rightarrow \frac{d}{dx} (\ln y) = \frac{d}{dx} (x \ln x) \]

  \[ \Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x + 1 \Rightarrow \frac{dy}{dx} = y(\ln x + 1) \Rightarrow (x^x)' = x^x(\ln x + 1). \]

- **Find** $\left((1 + \cos x)^{\sin x}\right)'$.
  **Solution.** Let $y = (1 + \cos x)^{\sin x}$. Then

  \[ \ln y = \ln((1 + \cos x)^{\sin x}) \Rightarrow \ln y = (\sin x) \ln(1 + \cos x) \]

  \[ \Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} ((\sin x) \ln(1 + \cos x)) \]

  \[ \Rightarrow \frac{dy}{dx} = (1 + \cos x)^{\sin x} \left( \cos x \ln(1 + \cos x) - \frac{(\sin x)^2}{1 + \cos x} \right). \]
Examples of Logarithmic Differentiation

- Find \( (x^x)' \).
  Solution. Let \( y = x^x \). Then
  \[
  \ln y = \ln(x^x) \Rightarrow \ln y = x \ln x \Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)
  \]
  \[
  \Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x + 1 \Rightarrow \frac{dy}{dx} = y(\ln x + 1) \Rightarrow (x^x)' = x^x(\ln x + 1).
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- Find \( ((1 + \cos x)^{\sin x})' \).
  Solution. Let \( y = (1 + \cos x)^{\sin x} \). Then
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  \[
  \Rightarrow \frac{dy}{dx} = (1 + \cos x)^{\sin x} \left( \cos x \ln(1 + \cos x) - \frac{(\sin x)^2}{1 + \cos x} \right).
  \]
Examples of Logarithmic Differentiation

Find \( \left( \frac{(x^2 + 1)^{2013}(x + 1)^{2014}}{(\cos x)^{2015}} \right)' \)

Solution. Let \( y = \frac{(x^2 + 1)^{2013}(x + 1)^{2014}}{(\cos x)^{2015}} \). Then

\[
\ln y = \ln \frac{(x^2 + 1)^{2013}(x + 1)^{2014}}{(\cos x)^{2015}}
\]

\[
\Rightarrow \ln y = 2013 \ln(x^2 + 1) + 2014 \ln(x + 1) - 2015 \ln \cos x
\]

\[
\Rightarrow \frac{dy}{dx} \ln y = \frac{4026x}{x^2 + 1} + \frac{2014}{x + 1} + 2015 \tan x
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{(x^2 + 1)^{2013}(x + 1)^{2014}}{(\cos x)^{2015}} \left( \frac{4026x}{x^2 + 1} + \frac{2014}{x + 1} + 2015 \tan x \right)
\]
Given a *word* problem about *related rates*, we need to do:

- interpret the problem correctly;
- establish the variables;
- find the relation of the variables \((F(x, y) = 0)\);
- determine which derivative \((dx/dt\) or \(dy/dt)\) is given and which derivative we are after;
- apply implicit differentiation to solve the problem.
Related Rates (Application of Implicit Differentiation)

Given a *word* problem about *related rates*, we need to do:

- interpret the problem correctly;
- establish the variables;
- find the relation of the variables \((F(x, y) = 0)\);
- determine which derivative \((dx/dt\) or \(dy/dt)\) is given and which derivative we are after;
- apply implicit differentiation to solve the problem.
Examples of Related Rates

Air is being pumped into the balloon at a rate of $a$ cm$^3$/s. How fast is the radius of the balloon increasing where the radius of the balloon is $b$ cm?

Air is being pumped into the balloon at a rate of $a$ cm$^3$/s. How fast is the surface area of the balloon increasing where the radius of the balloon is $b$ cm?
Examples of Related Rates

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- Air is being pumped into the balloon at a rate of $a$ cm$^3$/s. How fast is the surface area of the balloon increasing where the radius of the balloon is $b$ cm?