Limits

Notation:

\[
\frac{f(x_1) - f(x_0)}{x_1 - x_0} \to m \text{ as } x_1 \to x_0 \iff \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = m
\]

More generally,

\[
f(x) \to L \text{ as } x \to a \iff \lim_{x \to a} f(x) = L.
\]

“Definition” of Limits

Let \( f(x) \) be a function defined near \( a \). Then

\[
\lim_{x \to a} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) but not equal to \( a \).
Limits

Notation:

\[
\frac{f(x_1) - f(x_0)}{x_1 - x_0} \rightarrow m \text{ as } x_1 \rightarrow x_0 \iff \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = m
\]

More generally,

\[
f(x) \rightarrow L \text{ as } x \rightarrow a \iff \lim_{x \rightarrow a} f(x) = L.
\]

“Definition” of Limits

Let \( f(x) \) be a function defined near \( a \). Then

\[
\lim_{x \rightarrow a} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) but not equal to \( a \).
Real Definition of Limits

Let \( f(x) \) be a function defined near \( a \). Then

\[
\lim_{{x \to a}} f(x) = L
\]

if for every \( e > 0 \), we can find \( d > 0 \) such that

\[
|f(x) - L| < e
\]

for \( |x - a| < d \) and \( x \neq a \).
Examples of Limits

Explain why

$$\lim_{{x \to 1}} (x + 1) = 2$$

Numerical Evidence: As $x$ approaches 1, $f(x) = x + 1$ approaches 2:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 1$</td>
<td>2.1</td>
<td>2.01</td>
<td>2.001</td>
<td>1.9</td>
<td>1.99</td>
<td>1.999</td>
</tr>
</tbody>
</table>

Real Reason: The value of $x + 1$ can be made arbitrarily close to 2 by taking $x$ sufficiently close to 1 $\Leftrightarrow$

$$|f(x) - 2| = |(x + 1) - 2| < e$$
\text{as long as } |x - 1| < e

- $|x - 1| < .1 \Rightarrow |f(x) - 2| < .1$
- $|x - 1| < .01 \Rightarrow |f(x) - 2| < .01$
- $|x - 1| < .001 \Rightarrow |f(x) - 2| < .001$
Examples of Limits

Explain why

\[
\lim_{{x \to 1}} (x + 1) = 2
\]

Numerical Evidence: As \( x \) approaches 1, \( f(x) = x + 1 \) approaches 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 1 )</td>
<td>2.1</td>
<td>2.01</td>
<td>2.001</td>
<td>1.9</td>
<td>1.99</td>
<td>1.999</td>
</tr>
</tbody>
</table>

Real Reason: The value of \( x + 1 \) can be made arbitrarily close to 2 by taking \( x \) sufficiently close to 1 \( \iff \)

\[
|f(x) - 2| = |(x + 1) - 2| < \varepsilon \text{ as long as } |x - 1| < \varepsilon
\]

\[
\begin{align*}
| x - 1 | < .1 & \Rightarrow |f(x) - 2| < .1 \\
| x - 1 | < .01 & \Rightarrow |f(x) - 2| < .01 \\
| x - 1 | < .001 & \Rightarrow |f(x) - 2| < .001
\end{align*}
\]
Examples of Limits

Explain why

\[ \lim_{x \to 1} (x + 1) = 2 \]

Numerical Evidence: As \( x \) approaches 1, \( f(x) = x + 1 \) approaches 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 1 )</td>
<td>2.1</td>
<td>2.01</td>
<td>2.001</td>
<td>1.9</td>
<td>1.99</td>
<td>1.999</td>
</tr>
</tbody>
</table>

Real Reason: The value of \( x + 1 \) can be made arbitrarily close to 2 by taking \( x \) sufficiently close to 1 \( \iff \)

\[ |f(x) - 2| = |(x + 1) - 2| < e \text{ as long as } |x - 1| < e \]

- \( |x - 1| < .1 \iff |f(x) - 2| < .1 \)
- \( |x - 1| < .01 \iff |f(x) - 2| < .01 \)
- \( |x - 1| < .001 \iff |f(x) - 2| < .001 \)
Examples of Limits

Explain why

$$\lim_{{x \to 1}} (x + 1) = 2$$

Numerical Evidence: As $x$ approaches 1, $f(x) = x + 1$ approaches 2:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 1$</td>
<td>2.1</td>
<td>2.01</td>
<td>2.001</td>
<td>1.9</td>
<td>1.99</td>
<td>1.999</td>
</tr>
</tbody>
</table>

Real Reason: The value of $x + 1$ can be made arbitrarily close to 2 by taking $x$ sufficiently close to 1 $\iff$

$$|f(x) - 2| = |(x + 1) - 2| < e \text{ as long as } |x - 1| < e$$

- $|x - 1| < .1 \Rightarrow |f(x) - 2| < .1$
- $|x - 1| < .01 \Rightarrow |f(x) - 2| < .01$
- $|x - 1| < .001 \Rightarrow |f(x) - 2| < .001$
Examples of Limits

Explain why

\[ \lim_{{x \to 1}} (2x + 1) = 3 \]

Numerical Evidence: As \( x \) approaches 1, \( f(x) = x + 1 \) approaches 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x + 1 )</td>
<td>3.2</td>
<td>3.02</td>
<td>3.002</td>
<td>2.8</td>
<td>2.98</td>
<td>2.998</td>
</tr>
</tbody>
</table>

Real Reason: The value of \( 2x + 1 \) can be made arbitrarily close to 3 by taking \( x \) sufficiently close to 1 \( \Leftrightarrow \)

\[
|f(x) - 3| = |(2x + 1) - 3| < \varepsilon \text{ as long as } |x - 1| < \frac{\varepsilon}{2}
\]

\[ |x - 1| < .05 \Rightarrow |f(x) - 3| < .1 \]
\[ |x - 1| < .005 \Rightarrow |f(x) - 3| < .01 \]
\[ |x - 1| < .0005 \Rightarrow |f(x) - 3| < .001 \]
Examples of Limits

Explain why

\[ \lim_{x \to 1} (2x + 1) = 3 \]

Numerical Evidence: As \( x \) approaches 1, \( f(x) = x + 1 \) approaches 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x + 1 )</td>
<td>3.2</td>
<td>3.02</td>
<td>3.002</td>
<td>2.8</td>
<td>2.98</td>
<td>2.998</td>
</tr>
</tbody>
</table>

Real Reason: The value of \( 2x + 1 \) can be made arbitrarily close to 3 by taking \( x \) sufficiently close to 1 ⇔

\[ |f(x) - 3| = |(2x + 1) - 3| < \varepsilon \text{ as long as } |x - 1| < \frac{\varepsilon}{2} \]

- \( |x - 1| < .05 \Rightarrow |f(x) - 3| < .1 \)
- \( |x - 1| < .005 \Rightarrow |f(x) - 3| < .01 \)
- \( |x - 1| < .0005 \Rightarrow |f(x) - 3| < .001 \)
Examples of Limits

Explain why

\[ \lim_{x \to 1} (2x + 1) = 3 \]

Numerical Evidence: As \( x \) approaches 1, \( f(x) = x + 1 \) approaches 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x + 1 )</td>
<td>3.2</td>
<td>3.02</td>
<td>3.002</td>
<td>2.8</td>
<td>2.98</td>
<td>2.998</td>
</tr>
</tbody>
</table>

Real Reason: The value of \( 2x + 1 \) can be made arbitrarily close to 3 by taking \( x \) sufficiently close to 1 \( \iff \)

\[ |f(x) - 3| = |(2x + 1) - 3| < e \text{ as long as } |x - 1| < \frac{e}{2} \]

\[ |x - 1| < .05 \implies |f(x) - 3| < .1 \]
\[ |x - 1| < .005 \implies |f(x) - 3| < .01 \]
\[ |x - 1| < .0005 \implies |f(x) - 3| < .001 \]
Examples of Limits

Explain why

\[ \lim_{x \to 1} (2x + 1) = 3 \]

Numerical Evidence: As \( x \) approaches 1, \( f(x) = x + 1 \) approaches 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x + 1 )</td>
<td>3.2</td>
<td>3.02</td>
<td>3.002</td>
<td>2.8</td>
<td>2.98</td>
<td>2.998</td>
</tr>
</tbody>
</table>

Real Reason: The value of \( 2x + 1 \) can be made arbitrarily close to 3 by taking \( x \) sufficiently close to 1 \( \iff \)

\[ |f(x) - 3| = |(2x + 1) - 3| < e \text{ as long as } |x - 1| < \frac{e}{2} \]

\[ |x - 1| < .05 \implies |f(x) - 3| < .1 \]

\[ |x - 1| < .005 \implies |f(x) - 3| < .01 \]

\[ |x - 1| < .0005 \implies |f(x) - 3| < .001 \]
Examples and Comments

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

\[
\lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2
\]

\[
\lim_{x \to 0} (1 + x)^{1/x} = e
\]

\[
\lim f(x) \text{ is defined even if } f(a) \text{ is not defined.}
\]

\[
\lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a)
\]
Examples and Comments

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)
- \( \lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2 \)
- \( \lim_{x \to 0} (1 + x)^{1/x} = e \)
- \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.
- \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples and Comments

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)
- \( \lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2 \)
- \( \lim_{x \to 0} (1 + x)^{1/x} = e \)
- \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.
- \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples and Comments

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)
- \( \lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2 \)
- \( \lim_{x \to 0} (1 + x)^{1/x} = e \)
- \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.
- \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples and Comments

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

\[
\lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2
\]

\[
\lim_{x \to 0} (1 + x)^{1/x} = e
\]

- \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.
- \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples and Comments

• \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

• \( \lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2 \)

• \( \lim_{x \to 0} (1 + x)^{1/x} = e \)

• \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.

• \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples and Comments

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)
- \( \lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2 \)
- \( \lim_{x \to 0} (1 + x)^{1/x} = e \)
- \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.
- \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples and Comments

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)
- \( \lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2 \)
- \( \lim_{x \to 0} (1 + x)^{1/x} = e \)
- \( \lim_{x \to a} f(x) \) is defined even if \( f(a) \) is not defined.
- \( \lim_{x \to a} f(x) = \lim_{x \to 0} f(x + a) \)
Examples of Limits DNE

- \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist (DNE).
  
  As \( x \to 0 \), \( x^2 \to 0 \) and \( x^2 > 0 \). So \( \frac{1}{x^2} \) becomes arbitrarily large as \( x \to 0 \).

- \( \lim_{x \to 0} \frac{|x|}{x} \) DNE.
  
  \[
  \frac{|x|}{x} = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0 
  \end{cases}
  \]

- \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE
  
  Let \( x_n = 1/(2n\pi + \pi/2) \). Then \( \sin(1/x_n) = 1 \).
  Let \( x_n = 1/(2n\pi - \pi/2) \). Then \( \sin(1/x_n) = -1 \).
Examples of Limits DNE

- \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist (DNE).
  
  As \( x \to 0 \), \( x^2 \to 0 \) and \( x^2 > 0 \). So \( \frac{1}{x^2} \) becomes arbitrarily large as \( x \to 0 \).

- \( \lim_{x \to 0} \frac{|x|}{x} \) DNE.
  
  \[
  \frac{|x|}{x} = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0
  \end{cases}
  \]

- \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE
  
  Let \( x_n = \frac{1}{(2n\pi + \pi/2)} \). Then \( \sin(1/x_n) = 1 \).
  Let \( x_n = \frac{1}{(2n\pi - \pi/2)} \). Then \( \sin(1/x_n) = -1 \).
Examples of Limits DNE

- \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist (DNE).
  
  As \( x \to 0 \), \( x^2 \to 0 \) and \( x^2 > 0 \). So \( \frac{1}{x^2} \) becomes arbitrarily large as \( x \to 0 \).

- \( \lim_{x \to 0} \frac{|x|}{x} \) DNE.

  \[
  \frac{|x|}{x} = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0
  \end{cases}
  \]

- \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE
  
  Let \( x_n = 1/(2n\pi + \pi/2) \). Then \( \sin(1/x_n) = 1 \).
  
  Let \( x_n = 1/(2n\pi - \pi/2) \). Then \( \sin(1/x_n) = -1 \).
Examples of Limits DNE

- \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist (DNE).
  
  As \( x \to 0 \), \( x^2 \to 0 \) and \( x^2 > 0 \). So \( \frac{1}{x^2} \) becomes arbitrarily large as \( x \to 0 \).

- \( \lim_{x \to 0} \frac{|x|}{x} \) DNE.

\[
\frac{|x|}{x} = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0
\end{cases}
\]

- \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE

Let \( x_n = 1/(2n\pi + \pi/2) \). Then \( \sin(1/x_n) = 1 \).

Let \( x_n = 1/(2n\pi - \pi/2) \). Then \( \sin(1/x_n) = -1 \).
Examples of Limits DNE

- \[ \lim_{x \to 0} \frac{1}{x^2} \] does not exist (DNE).
  
  As \( x \to 0 \), \( x^2 \to 0 \) and \( x^2 > 0 \). So \( \frac{1}{x^2} \) becomes arbitrarily large as \( x \to 0 \).

- \[ \lim_{x \to 0} \frac{|x|}{x} \] DNE.

  \[
  \frac{|x|}{x} = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0 
  \end{cases}
  \]

- \[ \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \] DNE

  Let \( x_n = \frac{1}{(2n\pi + \pi/2)} \). Then \( \sin(1/x_n) = 1 \).
  
  Let \( x_n = \frac{1}{(2n\pi - \pi/2)} \). Then \( \sin(1/x_n) = -1 \).
Examples of Limits DNE

- \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist (DNE).
  
  As \( x \to 0 \), \( x^2 \to 0 \) and \( x^2 > 0 \). So \( \frac{1}{x^2} \) becomes arbitrarily large as \( x \to 0 \).

- \( \lim_{x \to 0} \frac{|x|}{x} \) DNE.
  
  \[
  \frac{|x|}{x} = \begin{cases} 
  1 & \text{if } x > 0 \\
  -1 & \text{if } x < 0 
  \end{cases}
  \]

- \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) DNE
  
  Let \( x_n = 1/(2n\pi + \pi/2) \). Then \( \sin(1/x_n) = 1 \).
  
  Let \( x_n = 1/(2n\pi - \pi/2) \). Then \( \sin(1/x_n) = -1 \).
“Definition” of Left-hand/Right-hand Limits

Let \( f(x) \) be a function defined in some interval \((a - r, a)\). Then

\[
\lim_{{x \to a^-}} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) but less than \( a \).

Let \( f(x) \) be a function defined in some interval \((a, a + r)\). Then

\[
\lim_{{x \to a^+}} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) but greater than \( a \).
“Definition” of Left-hand/Right-hand Limits

Let $f(x)$ be a function defined in some interval $(a - r, a)$. Then

$$\lim_{x \to a^-} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently close to $a$ but less than $a$.

Let $f(x)$ be a function defined in some interval $(a, a + r)$. Then

$$\lim_{x \to a^+} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently close to $a$ but greater than $a$. 
Examples and Comments

\[ \lim_{x \to 1^+} \frac{|x - 1|}{x - 1} \]

\[ \lim_{x \to 1^-} \frac{|x - 1|}{x - 1} \]

\[ \lim_{x \to 0^+} \lfloor x \rfloor \text{ where } \lfloor x \rfloor \text{ is the largest integer } \leq x. \]

\[ \lim_{x \to 0^-} \lfloor x \rfloor \]

\[ \lim_{x \to a} f(x) = L \text{ if and only if } \]

\[ \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \]
Examples and Comments

- \( \lim_{x \to 1^+} \frac{|x - 1|}{x - 1} \)
- \( \lim_{x \to 1^-} \frac{|x - 1|}{x - 1} \)
- \( \lim_{x \to 0^+} \lfloor x \rfloor \) where \( \lfloor x \rfloor \) is the largest integer \( \leq x \).
- \( \lim_{x \to 0^-} \lfloor x \rfloor \)
- \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \)
Examples and Comments

- \( \lim_{x \to 1^+} \frac{|x - 1|}{x - 1} \)
- \( \lim_{x \to 1^-} \frac{|x - 1|}{x - 1} \)
- \( \lim_{x \to 0^+} \lfloor x \rfloor \) where \( \lfloor x \rfloor \) is the largest integer \( \leq x \).
- \( \lim_{x \to 0^-} \lfloor x \rfloor \)
- \( \lim_{x \to a} f(x) = L \) if and only if

\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L
\]
Examples and Comments

- \( \lim_{x \to 1^+} \frac{|x - 1|}{x - 1} \)
- \( \lim_{x \to 1^-} \frac{|x - 1|}{x - 1} \)
- \( \lim_{x \to 0^+} \lfloor x \rfloor \) where \( \lfloor x \rfloor \) is the largest integer \( \leq x \).
- \( \lim_{x \to 0^-} \lfloor x \rfloor \)
- \( \lim_{x \to a} f(x) = L \) if and only if
  \[ \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \]
Examples and Comments

\[ \lim_{x \to 1^+} \frac{|x - 1|}{x - 1} \]

\[ \lim_{x \to 1^-} \frac{|x - 1|}{x - 1} \]

\[ \lim_{x \to 0^+} \lfloor x \rfloor \text{ where } \lfloor x \rfloor \text{ is the largest integer } \leq x. \]

\[ \lim_{x \to 0^-} \lfloor x \rfloor \]

\[ \lim_{x \to a} f(x) = L \text{ if and only if } \]

\[ \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \]
“Definition” of Infinite Limits

Let \( f(x) \) be a function defined near \( a \).

\[
\lim_{{x \to a}} f(x) = \infty
\]

if we can make the values of \( f(x) \) arbitrarily large positively by taking \( x \) sufficiently close to \( a \) but not equal to \( a \).

Let \( f(x) \) be a function defined near \( a \).

\[
\lim_{{x \to a}} f(x) = -\infty
\]

if we can make the values of \( f(x) \) arbitrarily large negatively by taking \( x \) sufficiently close to \( a \) but not equal to \( a \).
Infinite Limits

“Definition” of Infinite Limits

Let \( f(x) \) be a function defined near \( a \).

\[
\lim_{x \to a} f(x) = \infty
\]

if we can make the values of \( f(x) \) arbitrarily large positively by taking \( x \) sufficiently close to \( a \) but not equal to \( a \).

Let \( f(x) \) be a function defined near \( a \).

\[
\lim_{x \to a} f(x) = -\infty
\]

if we can make the values of \( f(x) \) arbitrarily large negatively by taking \( x \) sufficiently close to \( a \) but not equal to \( a \).
“Definition” of Infinite Limits

Let $f(x)$ be a function defined on $(b, \infty)$. Then

$$
\lim_{{x \to \infty}} f(x) = L
$$

if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently large positively.

Let $f(x)$ be a function defined on $(-\infty, b)$. Then

$$
\lim_{{x \to -\infty}} f(x) = L
$$

if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ sufficiently large negatively.
“Definition” of Infinite Limits

Let \( f(x) \) be a function defined on \((b, \infty)\). Then

\[
\lim_{x \to \infty} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently large positively.

Let \( f(x) \) be a function defined on \((-\infty, b)\). Then

\[
\lim_{x \to -\infty} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently large negatively.
Examples and Comments

- Explain \( \lim_{x \to 1^+} f(x) = -\infty \).
- Explain \( \lim_{x \to -\infty} f(x) = \infty \).
- Find the infinite limit \( \lim_{x \to 0^+} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to 0^-} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to \infty} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to -\infty} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to 0^+} \ln x \).
Examples and Comments

- Explain \( \lim_{x \to 1^+} f(x) = -\infty \).
- Explain \( \lim_{x \to -\infty} f(x) = \infty \).

- Find the infinite limit \( \lim_{x \to 0^+} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to 0^-} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to \infty} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to -\infty} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to 0^+} \ln x \).
Examples and Comments

- Explain \( \lim_{{x \to 1^+}} f(x) = -\infty \).
- Explain \( \lim_{{x \to -\infty}} f(x) = \infty \).

- Find the infinite limit \( \lim_{{x \to 0^+}} \frac{1}{x} \).
- Find the infinite limit \( \lim_{{x \to 0^-}} \frac{1}{x} \).
- Find the infinite limit \( \lim_{{x \to \infty}} \frac{1}{x} \).
- Find the infinite limit \( \lim_{{x \to -\infty}} \frac{1}{x} \).
- Find the infinite limit \( \lim_{{x \to 0^+}} \ln x \)
Examples and Comments

- Explain \( \lim_{x \to 1^+} f(x) = -\infty \).
- Explain \( \lim_{x \to -\infty} f(x) = \infty \).

Find the infinite limit \( \lim_{x \to 0^+} \frac{1}{x} \).

Find the infinite limit \( \lim_{x \to 0^-} \frac{1}{x} \).

Find the infinite limit \( \lim_{x \to \infty} \frac{1}{x} \).

Find the infinite limit \( \lim_{x \to -\infty} \frac{1}{x} \).

Find the infinite limit \( \lim_{x \to 0^+} \ln x \).
Examples and Comments

- Explain \( \lim_{x \to 1^+} f(x) = -\infty \).
- Explain \( \lim_{x \to -\infty} f(x) = \infty \).
- Find the infinite limit \( \lim_{x \to 0^+} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to 0^-} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to \infty} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to -\infty} \frac{1}{x} \).
- Find the infinite limit \( \lim_{x \to 0^+} \ln x \).
Examples and Comments

- Explain \( \lim_{x \to 1^+} f(x) = -\infty \).
- Explain \( \lim_{x \to -\infty} f(x) = \infty \).

Find the infinite limit:
- \( \lim_{x \to 0^+} \frac{1}{x} \)
- \( \lim_{x \to 0^-} \frac{1}{x} \)
- \( \lim_{x \to \infty} \frac{1}{x} \)
- \( \lim_{x \to -\infty} \frac{1}{x} \)
- \( \lim_{x \to 0^+} \ln x \)
Examples and Comments

- Explain $\lim_{x \to 1^+} f(x) = -\infty$.
- Explain $\lim_{x \to -\infty} f(x) = \infty$.
- Find the infinite limit $\lim_{x \to 0^+} \frac{1}{x}$.
- Find the infinite limit $\lim_{x \to 0^-} \frac{1}{x}$.
- Find the infinite limit $\lim_{x \to \infty} \frac{1}{x}$.
- Find the infinite limit $\lim_{x \to -\infty} \frac{1}{x}$.
- Find the infinite limit $\lim_{x \to 0^+} \ln x$. 
Examples and Comments

- \( \lim_{x \to \infty} f(x) = \lim_{x \to 0^+} f(\frac{1}{x}) \)
- \( \lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f(\frac{1}{x}) \)
- If \( \lim_{x \to a} f(x) = \infty \) or \( \lim_{x \to a} f(x) = -\infty \)
  then \( \lim_{x \to a} \frac{1}{f(x)} = 0. \)
Examples and Comments

- \( \lim_{x \to \infty} f(x) = \lim_{x \to 0^+} f\left(\frac{1}{x}\right) \)

- \( \lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f\left(\frac{1}{x}\right) \)

- If \( \lim_{x \to a} f(x) = \infty \) or \( \lim_{x \to a} f(x) = -\infty \)

then

\( \lim_{x \to a} \frac{1}{f(x)} = 0. \)
Examples and Comments

\[ \lim_{x \to \infty} f(x) = \lim_{x \to 0^+} f\left(\frac{1}{x}\right) \]

\[ \lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f\left(\frac{1}{x}\right) \]

If

\[ \lim_{x \to a} f(x) = \infty \text{ or } \lim_{x \to a} f(x) = -\infty \]

then

\[ \lim_{x \to a} \frac{1}{f(x)} = 0. \]