Solutions for Math 113/114 H1 Final 2014 Fall

Multiple Choices:

V1. DACBBBCCDA

V2. ACDBBBCDAC

1. (15 points) Evaluate the following integrals:

(a) (5 points) \( \int_0^3 |x^2 - 3x + 2| \, dx \)

Solution. Since

\[
\begin{cases}
  x^2 - 3x + 2 \geq 0 & \text{for } x \in [0, 1] \cup [2, 3] \\
  x^2 - 3x + 2 \leq 0 & \text{for } x \in [1, 2],
\end{cases}
\]

\[
\int_0^3 |x^2 - 3x + 2| \, dx = \int_0^1 (x^2 - 3x + 2) \, dx - \int_1^2 (x^2 - 3x + 2) \, dx + \int_2^3 (x^2 - 3x + 2) \, dx
\]

\[
= \left( \frac{x^3}{3} - \frac{3}{2} x^2 + 2x \right)_0^1 - \left( \frac{x^3}{3} - \frac{3}{2} x^2 + 2x \right)_1^2 + \left( \frac{x^3}{3} - \frac{3}{2} x^2 + 2x \right)_2^3
\]

\[
= \left( \frac{5}{6} - \left( -\frac{1}{6} \right) \right) + \frac{5}{6} = \frac{11}{6}
\]

(b) (5 points) \( \int_0^1 \frac{e^x}{e^x + 2} \, dx \)

Solution. Substitute \( t = e^x + 2 \):

\[
\int_0^1 \frac{e^x}{e^x + 2} \, dx = \frac{t \, dt}{e^x dx} \int_3^{2+e} \frac{dt}{t} = \ln |t|_3^{2+e} = \ln(2 + e) - \ln 3
\]

\[\text{http://www.math.ualberta.ca/~xichen/math11314f/113H1_FINAL-2014_FALL-SOLUTIONS.pdf}\]
(c) (5 points) \( \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx \)

Solution. Substitute \( t = \sqrt{x} \):

\[
\int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx = \int \frac{\cos(t)}{t} \, dt = 2 \int \cos t \, dt = 2 \sin t + C = 2 \sin(\sqrt{x}) + C
\]

2. (20 points) Let \( f(x) = \frac{2 + x - x^2}{(x - 1)^2} \). Given that \( f'(x) = \frac{x - 5}{(x - 1)^3} \) and \( f''(x) = \frac{14 - 2x}{(x - 1)^4} \), find each of the following:

(a) (2 points) The domain of \( f \) and intercepts with \( x \) and \( y \) axes.
(b) (4 points) Vertical and horizontal asymptotes.
(c) (4 points) The intervals of increase and decrease.
(d) (4 points) The local maximum and minimum values.
(e) (4 points) The intervals of concavity and inflection points.
(f) (2 points) Sketch the graph of \( f \).

Show all your work and justify each answer.

Solution. (a) The domain of \( f(x) \) is \( \{ x \neq 1 \} = (-\infty, 1) \cup (1, \infty) \).

Solving \( 2 + x - x^2 = 0 \), we obtain \( x = -1 \) and \( x = 2 \). So \( y = f(x) \) has two \( x \)-intercepts: \((-1, 0)\) and \((2, 0)\). And it has one \( y \)-intercept \((0, f(0)) = (0, 2)\).

(b) Since

\[
\lim_{x \to 1} \frac{2 + x - x^2}{(x - 1)^2} = \infty
\]

\( y = f(x) \) has a vertical asymptote \( x = 1 \). Since

\[
\lim_{x \to \infty} \frac{2 + x - x^2}{(x - 1)^2} = \lim_{x \to \infty} \frac{2}{x^2} + \frac{1}{x} - \frac{1}{1} = -1
\]

\[
\lim_{x \to -\infty} \frac{2 + x - x^2}{(x - 1)^2} = \lim_{x \to -\infty} \frac{2}{x^2} + \frac{1}{x} - \frac{1}{1} = -1
\]
\( y = f(x) \) has a horizontal asymptote \( y = -1 \).

(c) Since \( f'(x) > 0 \) for \( x > 5 \) or \( x < 1 \) and \( f'(x) < 0 \) for \( 1 < x < 5 \),
\( f(x) \) is increasing on \((-\infty, 1)\) and \([5, \infty)\) and decreasing on \((1, 5]\).

(d) Since \( f'(x) \) changes from negative to positive at 5, \( f(x) \) has a local minimum \( f(5) = -9/8 \) at \( x = 5 \).

(e) Since \( f''(x) > 0 \) for \( x < 7 \) and \( f''(x) < 0 \) for \( x > 7 \), \( f(x) \) is concave upward on \((-\infty, 1)\) and \((1, 7)\) and concave downward on \((7, \infty)\). It has an inflection point at \((7, -10/9)\).

(f)
3. (10 points) A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the ground changing when 200 ft of string has been let out?

Solution. As in the figure above, $K$ is the position of the kite and $G$ is the ground end of the string. Let $x$ be the angle between the string and the ground (in rad), $y$ be the horizontal distance between the end of the string and the kite (in ft) and $t$ be the time variable (in s). Then

$$\cot x = \frac{y}{100} \Rightarrow \frac{d}{dt} \cot x = \frac{d}{dt} \left( \frac{y}{100} \right) \Rightarrow - \csc^2 x \frac{dx}{dt} = \frac{1}{100} \frac{dy}{dt}$$

$$\Rightarrow \frac{dx}{dt} = - \frac{\sin^2 x \frac{dy}{dt}}{100}$$

When the length of the string is $|GK| = 200$, $\sin x = 100/200 = 1/2$ and

$$\frac{dx}{dt} = - \left( \frac{(1/2)^2}{100} \right) 8 = - \frac{1}{50}.$$  

So the angle is decreasing at a rate of $1/50$ rad/s.

4. (10 points) You need to fence in a rectangular play zone for children to fit into a right-triangular plot with sides measuring 4 m and 12 m. Two sides of the rectangle should be on the sides and one vertex on the hypotenuse of the triangle. What is the maximum area for this play zone?
Solution. Let $x$ and $y$ be the length and width of the rectangle as in the figure (in m). Then the area of the rectangle is $A = xy$ and

$$\frac{x}{12} = \frac{4 - y}{4} \Rightarrow y = 4 - \frac{x}{3}$$

since $\triangle ABC$ and $\triangle AB'C'$ are similar. Therefore,

$$A = x(4 - \frac{x}{3}) = 4x - \frac{x^2}{3} = f(x)$$

and it suffices to find the maximum of $f(x)$ on $[0, 12]$:

- Solve $f'(x) = 0$:

  $$f'(x) = 0 \Rightarrow 4 - \frac{2x}{3} = 0 \Rightarrow x = 6.$$ 

- $f(6) = 12$.
- $f(0) = f(12) = 0$.
- Comparing $f(6), f(0)$ and $f(12)$, we conclude that $f(x)$ takes the maximum 12 when $x = 6$.

So the maximum area of the rectangular play zone is 12 m$^2$ when the play zone has dimension 6m $\times$ 2m.

5. (15 points) Do the following:
(a) (7 points) Evaluate the integral \( \int_0^3 \left( \frac{x^2}{9} + 1 \right) \, dx \) using the definition of definite integrals. Use right end points. No marks will be given if the definition is not used.

\[ \sum_{i=1}^n i^2 = \frac{n(n + 1)(2n + 1)}{6}. \]

**Solution.** Using right endpoint approximation, let \( \Delta x = \frac{3}{n} \) and \( x_i = x_0 + i\Delta x = 3i/n \). Then

\[
\int_0^3 \left( \frac{x^2}{9} + 1 \right) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x
= \lim_{n \to \infty} \sum_{i=1}^n \frac{3}{n} \left( \frac{1}{9} \left( \frac{3i}{n} \right)^2 + 1 \right)
= \lim_{n \to \infty} \left( \frac{3}{n^3} \sum_{i=1}^n i^2 + \frac{3}{n} \sum_{i=1}^n 1 \right)
= \lim_{n \to \infty} \left( \frac{3}{n^3} \left( \frac{n(n + 1)(2n + 1)}{6} \right) + 3 \right)
= \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) \left( 1 + \frac{1}{2n} \right) + 3 = 4.
\]

\( \square \)

(b) (2 points) Evaluate the integral \( \int_0^3 \left( \frac{x^2}{9} + 1 \right) \, dx \) using the Fundamental Theorem of Calculus.

**Solution.** Using FTC, we have

\[
\int_0^3 \left( \frac{x^2}{9} + 1 \right) \, dx = \left[ \frac{x^3}{27} + x \right]_0^3 = 4
\]

\( \square \)

(c) (6 points) Show that the equation \( x^5 + 2x + 1 = 0 \) has exactly one solution.

**Proof.** Let \( f(x) = x^5 + 2x + 1 \). Clearly, \( f(x) \) is differentiable on \( (-\infty, \infty) \) and \( f'(x) = 5x^4 + 2 \).

6
Since $f(-1) = -2 < 0$ and $f(0) = 1 > 0$, $f(x_0) = 0$ for some $x_0$ in $(-1, 0)$ by Intermediate Value Theorem. So $f(x) = 0$ has at least one solution in $(-1, 0)$.

If $f(x) = 0$ has two solutions $f(x_1) = f(x_2) = 0$ for some $x_1 < x_2$, then

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

for some $x_3$ in $(x_1, x_2)$ by Mean Value Theorem. But

$$f'(x) = 5x^4 + 2 > 0$$

for all $x$.

Contradiction. So $f(x) = 0$ has no more than one solution.

In conclusion, $f(x) = 0$ has exactly one solution. \qed