1. Project Summary

Two cities $A$ and $B$ are connected by a network of highways as

![Network Diagram]

During a heavy snow storm, each highway (represented by an edge) has an independent chance of 50 percent of being closed. Find the probability that one can still drive from $A$ to $B$.

This project requires some basic knowledge of graph theory and probability theory.

2. Network Outage

We may ask a more general question.

**Question 2.1.** Given a network of nodes, i.e., a graph, we know that every edge of the graph has a fixed independent chance of being broken. What is the probability that the network remains connected? We call this number the survival rate of the network.

We can represent such a network by a weighted graph $G$ with the weight of an edge being the probability of this edge NOT being broken. We use the notation $s(G)$ for the probability of $G$ remaining connected. Furthermore, we can fix two nodes $A$ and $B$ of $G$ and let $s(G, A, B)$ be the probability of $A$ and $B$ remaining connected by a path.

Here we allow $G$ to have multiple edges.

We start with a couple of examples and observations.
Example 2.2. If $G$ is a tree, breaking any edge of $G$ will render $G$ disconnected. Therefore,

$$(2.1)\quad s(G) = w_1w_2...w_n$$

where $w_1, w_2, ..., w_n$ are the weights of all edges in $G$. Let $A$ and $B$ be two nodes $G$. Then $A$ and $B$ are joined by a unique path $P$ in $G$ and $A$ and $B$ remains connected if and only if no edges of $P$ are broken. Therefore,

$$(2.2)\quad s(G, A, B) = w_1w_2...w_m$$

where $w_1, w_2, ..., w_m$ are the weights of all edges in $P$.

Example 2.3. If $G$ consists of only two nodes $A$ and $B$ with $n$ edges joining them, then

$$(2.3)\quad s(G) = 1 - (1 - w_1)(1 - w_2)...(1 - w_n)$$

where $w_1, w_2, ..., w_n$ are the weights of all edges joining $A$ and $B$. More generally, if $G$ contains two nodes $A$ and $B$ with $n$ edges joining them of weights $w_1, w_2, ..., w_n$, we can replace the $n$ edges by one edge of weight given in (2.3); $s(G)$ remains the same.

3. Recursion Algorithm

Let $G$ be a weighted graph and $e$ be an edge joining two nodes $U$ and $V$ of weight $w$. Suppose that there are no other edges between $U$ and $V$.

We can have either $e$ breaks or $e$ does not.

If $e$ breaks, let $G - e$ be the graph of $G$ with only the edge $e$ removed.

If $e$ does not break, let $G/e$ be the graph of $G$ obtained by contracting $e$ to one node, say, $W$ and replacing every edge joining a node $A$ to one of $U$ and $V$ by an edge of the same weight joining $A$ to $W$.

Then

$$(3.1)\quad s(G) = (1 - w)s(G - e) + ws(G/e).$$

If we fix two nodes $A$ and $B$ of $G$, we have

$$(3.2)\quad s(G, A, B) = (1 - w)s(G - e, A, B) + ws(G/e, A, B).$$

Example 3.1. Consider $s(G)$ for the weighted triangle $G$ (choose $e = AB$)
Then
\[ s(G) = (1 - \frac{1}{2})s(G - e) + \frac{1}{2} s(G/e) \]
\[ = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^2 + \frac{1}{2} \left( 1 - \left( \frac{1}{2} \right)^2 \right) = \frac{1}{2} \]
where \( s(G - e) \) is computed using Example 2.2 and \( s(G/e) \) is computed using Example 2.3.

**Example 3.2.** Consider \( s(G, A, D) \) for (choose \( e = AB \))

Here we can use Example 2.3 to replace the multiple edge in \( G/e \):

\[ G/e \]

Then
\[ s(G, A, D) = (1 - \frac{1}{2})s(G - e, A, D) + \frac{1}{2} s(G/e, A, D) \]
\[ = \frac{1}{2} \left( \frac{1}{2} \right)^2 + \frac{1}{2} \left( \frac{3}{4} \right) \left( \frac{1}{2} \right) = \frac{5}{16} \]
where both \( s(G - e, A, D) \) and \( s(G/e, A, D) \) are computed using Example 2.2.

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