APPENDIX 2
Maple® V Labs
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Lab 1. Introduction to MAPLE® V.

In this Lab, we present a short introduction to Maple, including only the basic syntax structure and capabilities. For a complete tutorial, see First Leaves: A Tutorial Introduction to Maple V, Maple for the Calculus Student: A Tutorial, or The Maple V Flight Manual. For information about Maple, send e-mail to info@maplesoft.com. To access Maple information on the World Wide Web, set your browser to visit the URL http://www.maplesoft.com.

1. Getting Started with Maple
After starting the program and beginning a new worksheet, Maple awaits your instructions. A prompt symbol “>” is displayed on the screen, followed by the input bar, a vertical bar “|” that indicates where the characters you type will be inserted. You must mark the end of a Maple statement by typing a semicolon “;” or colon “:”. Press the ENTER key to have Maple perform your command. Try:

> 2+3;

Try again with a colon “:” instead of a semicolon “;”: 

> 2+3:

Maple did add 2 and 3, but the colon indicates that you didn’t want to be bothered with the result.

To leave the program, either choose Quit from the file menu or enter the command (Note: if you do not want to quit Maple, please do not hit the ENTER key after this command!):

> quit

This is one of the very rare commands that doesn’t require a terminating semicolon or colon.

2. Online Help
There are two methods to get online help in Maple. The first is the Help Browser. Choose Index from the Help menu. Click on items in the windows to get information. The second method is to enter a help statement directly into your worksheet. There
are three help operators in Maple: ?, ??, ???. The help operators, like quit, do not need a trailing semicolon. To see the full help screen for factor, try:

```
> ? factor
```

Try getting help for help with:

```
> ? ?
```

The second help operator, ??, returns the calling sequence—that is, the syntax for the command. Try:

```
> ?? factor
```

The third operator, ???, show examples, which is very useful:

```
> ??? factor
```

If Maple doesn’t recognize what you asked, it offers suggestions. Try:

```
> ? fact
```

Also, you can type help(topic) to get the information for the topic, but this command should end up with a semicolon.

```
> help(factor);
> help(fact);
```

3. Some Maple Characters, Commands and Their Meanings

Maple uses what is called infinite precision arithmetic. This means that all calculations are exact (up to machine and memory limits). Maple uses the standard symbols for arithmetic: + (addition), - (subtraction), * (multiplication), / (division), ^ (exponentiation).

```
> (3+4!)^2-13;
> 7*(3+5)-(45-15)/6+13;
> sqrt(%); #% is short for 'the last expression'
```

The \texttt{sqrt(x)} takes the positive square root of \texttt{x}.

There are two ways to write down your comments. The first is already shown in the above: in Maple input status, put the pound character \# before the comments. Also, you can click the button T from the tool bar to write down the text comments like the following.

```
> sqrt(%);
> % is short for 'the last expression'
```

The command \texttt{abs( )} takes the absolute value of the inside expression.

```
> abs(-6);
```

In addition to arithmetic, Maple does algebra in a symbolic form.

```
> abs(x);
> absolute value of x or an expression
> sqrt(x);
```
You can assign some expressions to some variables (with names) using “:=”, do some symbolic computation, such as factor, expand, simplify, all the arithmetic characters +, -, *, /, ^, also work.

\[
\begin{align*}
  > f1 & := x^2 + 4*x - 5; \\
  > f2 & := \text{factor}(f1); \\
  > f3 & := \text{expand}(f2) + 9; \\
  > f4 & := f3/(x+2); \\
  > f5 & := \text{simplify}(f4);
\end{align*}
\]

To insert a particular value into \( x \)'s in the \( f3 \) expression, we use \texttt{subs( )} command which performs substitution or replacement.

\[
\begin{align*}
  > \text{subs}(x=3,f3);
\end{align*}
\]

Solve the equations for a certain variable.

\[
\begin{align*}
  > \text{solve}(f1=0,x); \\
  > \text{solve}(f1=-10,x);
\end{align*}
\]

Notice the symbol \texttt{I} appears in the answer. In Maple, \texttt{I} is the complex number \( i \). When the \texttt{I} symbol appears, the solutions are not real numbers, since no real number equals the square root of \( -1 \). The symbol \texttt{I} denotes one of the complex roots of \(-1\).

\[
\begin{align*}
  > \text{solve}(x^4+x^2+x+1=0);
\end{align*}
\]

Hit the \texttt{ENTER} key to execute the above Maple command. The answer says "RootOf...". It’s Maple’s way of saying it doesn’t have the ability to find the roots exactly.

It can estimate the root numerically, but we won’t pursue this topic now (see Lab2). Instead, we can also use a graphical method to find roots approximately (details in Lab3). First of all, find out how to use the \texttt{plot( )} command in Maple.

Plot an expression \( f1 \) for a variable \( x \), plotting the graph on the default interval \([-10, 10]\).

\[
\begin{align*}
  > \text{plot}(f1,x);
\end{align*}
\]

We can specify the domain of \( x \) as following. Notice that we use two dots when doing so: \( x=-6..4 \). Also in the same way we specify the range of \( y \).

\[
\begin{align*}
  > \text{plot}(f1,x=-6..4); \\
  > \text{plot}(f1,x=-5..5,y=-4..4);
\end{align*}
\]

Furthermore, we can plot many functions on a single set axes. Simply enclose all the functions in an extra set of curly braces \{ \}.

\[
\begin{align*}
  > \text{plot}\{f1,f5\},x,y=-8..8;
\end{align*}
\]

4. Exercises:

\begin{itemize}
  \item 1. Multiply \((3x - y + 5z)^3(x + 3y - z)^2\).  
\end{itemize}
2. Find the factors of \( 2x^6 + 3x^5 - 63x^4 - 55x^3 + 657x^2 + 216x - 2160. \)

3. Define \( f_1 = 4999x^6 + 12124757x^4 - 19683x^2 - 531441. \) Plot the graph of \( f_1. \)

4. Plot the graph \( f_1 = \frac{2}{3x} + x \) on the intervals \([-5, 5], [-1, 1], [0, 1]\). What is the domain of \( f(x) = f_1? \)

5. Solve the equation \( |2x - 1| = 9 \) for the exact solution. Draw a graph, can you find the roots approximately?

5. Supplementary Exercises

1. Compute \( 2 \times 10^8 \times 3 \times 12 / (6^6 \times 10!) \) and \( 1/2 + 0.5.\)

2. Solve equation \( x^2 + x - 6 = 0. \)

3. Simplify the expression \( \frac{x^2 - 1}{(x - 1)^2}. \)

4. Define \( f_1 = 3x - 4, \) plot the graph of \( f_1, \) solve \( f_1=0. \)

5. Solve \( |4x - 5| = |1 + x|, \) check the answer graphically.

6. Solve the equations: \( 2x + 3y = 4, 3x - 2y = -1 \) for the variables \( x \) and \( y. \)

7. Define \( A = x^2 + 1, B = x^2 - 1, \) expand the multiplication \( AB. \)

8. Define \( expr = x^2 + 3x - 18, \) compute the value of \( expr \) when \( x=2, \) factor \( expr, \) compute the \( expr \) when \( x=a+h, \) plot \( expr \) for \( x \) on the interval \([-5, 5].\)

9. Using \( ? \) to get the online help for plot, try \( plot, options. \) There are also extensive three-dimensional capabilities, see \( ?plot3d, ?plot3d, options. \)

Lab 2. Numbers

In this lab, we will explore all the numbers: integers; rationals, irrationals; real numbers, complex numbers. Also, we include equations solving which is related to numbers.
1. Real Numbers
Like any other calculators, Maple does the arithmetic on integer numbers, fractions which are rational numbers and on decimal numbers.

\[
\begin{align*}
&5+3; \\
&(2-7.3)/5.9;
\end{align*}
\]

Maple calls decimal numbers **floating point numbers**. By default, Maple displays 10 digits of accuracy for floating point numbers. For decimal values, Maple uses approximate arithmetic. (The number of digits is set with `Digits`; see ?Digits.)

Maple does mathematics numerically, as well as symbolically. Because of the symbolic computation, Maple uses infinite precision arithmetic, which means all calculations are exact up to machine and memory limits. Also because of the symbolic computation, Maple can display fractions, irrational numbers, as well as complex numbers.

\[
\begin{align*}
&1/2+2/3; \\
&\text{fraction number, rational number} \\
&\sqrt{2}; \\
&\text{irrational number} \\
&3^{1/3}; \\
&\text{irrational number} \\
&\sqrt{3} + 3\sqrt{5} - 4\sqrt{3};
\end{align*}
\]

Maple will not convert fractions to decimal numbers unless you ask it to.

\[
\begin{align*}
&1./2+2/3;
\end{align*}
\]

In the above command, 1. is a floating point number, so Maple automatically treats 1./2+2/3 as a floating point number. The function that converts from rational to floating point is `evalf`. (see ?evalf).

\[
\begin{align*}
&\text{evalf}(1/2+1/3); \\
&\text{evalf}(1/2+1/3, 15); \\
&\text{evalf}(\sqrt{2}); \\
&\text{evalf}(\sqrt{2}, 25);
\end{align*}
\]

Using this command, it’s easy to verify the decimal periods for rational numbers and irrational numbers.

We can name objects in Maple for later use as we name a variable in mathematics. Use the form `name:=value`. Names are case-sensitive; that is, `S1` is different from `s1`.

\[
\begin{align*}
&q:=\sqrt{5}; \\
&\text{evalf}(q); \\
&\text{evalf}(q, 60); \\
&\text{evalf}(2/3+1/719, 15);
\end{align*}
\]

In this interface, to change an earlier input to Maple, simply use the mouse to insert the cursor (by clicking) at the point of correction or alteration, and insert or delete
characters as needed. Hitting the execution key (ENTER key) will reexecute a Maple command, replacing old input with new. Since input and output can be selected with the mouse and deleted, it is possible to create unfathomable worksheets, and worksheets with surprises. For example, if the input q:=sqrt(5); is deleted from the worksheet, the variable q still has the value of √5. Cutting from the worksheet does not change the memory state. To free a name from a definition, give it the value of its own name (in single quota).

> q:='q';
> q;

Maple also knows several constants like π (see ? constants).

> Pi;
> evalf(Pi,100);

If the letters pi are used in a Maple input, Maple will return the symbol π because Maple returns Greek symbols for all the Greek alphabet. However, pi unlike Pi, is not known to Maple as the ratio between the circumference and the diameter of a circle.

> pi; beta; nu; eta; theta; omega; gamma; Omega; lambda; Lambda;

2. Complex numbers
Recall in the previous lab, some roots of the equation cannot be represented as real numbers, the symbol I appears. I is very special in Maple, which represents the number i. There are real part and imaginary part in every complex number.

> sqrt(-1);
> 1+3*I;
> Re(1+3*I);
> Im(1+3*I);

Maple does all the arithmetic computation with complex numbers, as well as the conjugate, norm (by abs( )) etc.

> z1:=1+3*I;
> z2:=3.1-5.2*I;
> conjugate(z1); abs(z2);
> z1*conjugate(z1); abs(z1)^2;

With this, we can verify all the properties of complex numbers. As in the following, we verify \( \overline{z_1 z_2} = \overline{z_1} \overline{z_2} \).

> conjugate(z1*z2);
> conjugate(z1)*conjugate(z2);

Last, we mention three percentage signs which are used to refer to previously computed expressions. The % operator reevaluates the last expression computed, the
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%% operator reevaluates the second last expression computed, and the %%% operator reevaluates the third last expression computed. To see the details for these operators, check ?ditto.

\begin{verbatim}
> a; b; c;
> %+%+%%%;
\end{verbatim}

3. Solving equations
Maple also does arithmetic with equations. You can add or subtract pairs of equations. Equations can also be multiplied or divided by expressions.

\begin{verbatim}
> 2*x+3*y=1;
> eq1:=2*x+3*y=1;
> eq2:=-2*x=-2*x;
> eq1+eq2;
> 1/3*(%);
\end{verbatim}

Of course, equations can be solved for a variable. (See ? solve.) For example, solve the equation \( x^3 - 8x + 10 = 0 \).
Define the equation and assign it to the variable eq1

\begin{verbatim}
> eq1:=x^3-8*x+10=0;
\end{verbatim}

Solve the equation eq1 for x and assign the results to an1

\begin{verbatim}
> an1:=solve(eq1,x);
\end{verbatim}

Symbolic solutions can get cumbersome! Obtain a floating point equivalent of these roots by evaluating the results to floating point numbers.

\begin{verbatim}
> evalf(an1);
\end{verbatim}

Notice that there are two complex roots. We can reference an individual root by the selector notation.

\begin{verbatim}
> an1[1];
> evalf(an1[1]);
\end{verbatim}

Sometimes it is necessary to solve an equation numerically.

\begin{verbatim}
> fsolve(eq1,x);
\end{verbatim}

The floating point solver obtained just the real root. But it can be instructed to obtain all the roots.

\begin{verbatim}
> fsolve(eq1,x,complex);
\end{verbatim}

Plot the graph of the relation \( x^2 + y^2 = 4 \). This is a circle that represents a function \( y = y(x) \) implicitly. Hence, we can either use the implicit plot command (which will be discussed later) or we can solve for \( y(x) \) explicitly. Here, we will follow the second alternative.

\begin{verbatim}
> q:=solve(x^2+y^2=4,y);
> plot({q},x=-2..2,scaling=constrained);
\end{verbatim}
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constrained causes both axes to have equal scales. It could be invoked interactively from the options window once the user saw the graph of the ellipse where that of a circle we expected.

4. Exercises

• 1. Find \( \frac{1}{3} + \frac{1}{7} \) to 20 decimal places.

• 2. Find a five decimal-place approximation to \( \pi x^2 \) when a) \( x=2 \); b) \( x=3.5 \).

• 3. Use Maple to verify all the properties of real numbers, complex numbers.

• 4. In the same window, plot the graphs of \( y = 3 - x \) and \( y = x^2 - 2 \), find an intersection point of the line and the parabola.

• 5. Find the real roots of the cubic equation \( x^3 - x^2 - 100x + 310 = 0 \).

5. Supplementary Exercises

• 1. Find \( \frac{1}{17} + \frac{1}{177} \) to 15 decimal places.

• 2. Use your favorite technique to find the coordinates of the vertex of the parabola given by \( y = 10x^2 + 26x - 309 \).

• 3. Plot the upper branch of the parabola \( x = \frac{y^2}{2} - 1 \).

• 4. Plot the graph of the line that passes through the point (3,1) and is parallel to the line \( 3x + 2y = 10 \).

• 5. A circle in quadrant II is tangent to both axes, and the point (-8,9) is on the circle, plot the circle.
• 6. Produce a plot or a hand drawn graph supported by Maple plots that gives a representative image of the graph of the cubic polynomial $y = x^3 - x^2 - 100x + 310$.

• 7. Plot the graph of $4x^2 + 9y^2 = 13$.

• 8. A circle in quadrant I is tangent to both axes, and the point (2,0) is on the circle. Plot the circle.

• 9. Define $f(x) = 2x^5 - 25x^2 + 7$, solve each of the following for $x$: a) $fx = 0$; b) $fx > 5$; c) $1 < fx < 2$.

**Lab 3. Study of Functions**

In this Lab we present an introduction to functions in Maple and we will familiarize you with function definition and function plotting. It only gives you some general and basic ideas of function manipulation in Maple. More of function studies will come in later Labs.

1. **Defining Functions**

From the previous Labs, we know how to define an expression:

```maple
> e1:=x^3+3*x;
```

To evaluate it at $x = \frac{1}{2}$, we use the `subs()` command:

```maple
> subs(x=1/2,e1);
```

To get a decimal answer, we must use `evalf()`:

```maple
> evalf(%);
```

Maple defines functions by using the arrow notation. To make it, first type a dash - and then follow it with a greater than symbol $>$ to get $->$. Define the following function:

```maple
> f:=x->x^3+3*x;
```

This notation says that $f$ is the function that takes $x$ as input, computes $x^3 + 3x$, and returns the result as output. To evaluate a function, type:

```maple
> f(1/2);
```

2. **Ploting Functions**
The plotting commands for functions are slightly different than those for expression (introduced in lab1). To find the difference, let’s take the same example: plot the function $x^2 + 4x - 5$. First define the function by using arrow, then plot the graph of this function without specifying a domain and range.

```
> f:=x->x^2+4*x-5;
> plot(f);
```

The default interval for $x$ is $[-10,10]$. We can specify the domain of $x$ as following. Notice that we use two dots when doing so: $x=-6..4$. Also in the same way we specify the range of $y$.

```
> plot(f,-5..5);
> plot(f,-5..5,-4..4);
```

Plots $f$ on the interval $[-5,5]$, range $[-4,4]$. With the above graph of $f$, we can solve the equation $f(x)=0$ numerically. It is necessary to vary the domain and range to improve your plots. Change the plot domain and/or range to zero in those points where the graph crosses the $x$-axis. Click on the graph to select it. Use the mouse to place the cursor at each of the points where the curve crosses the axis. Depress the mouse button. Maple now displays the coordinates of the cursor’s position. Look for this display on the far left of the menu bar. This technique helps you accurately locate points. When you use it, remember to click twice-once to select the graph and then once more when you have located the point whose coordinates you wish to know. Knowing these coordinates, you should now be able to say for what values of $x$ the equation $f(x)=0$ is true, also for what values of $x$ the inequality $f(x)>0$ is true.

For an expression, there is a little bit different.

```
> f1:=f(x);
> plot(f1,x);
> plot(f1,x=-5..5);
> plot(f1,x=-5..5,y=-4..4);
```

Notice the command syntax for expression: $x=-5..5$ instead of $-5..5$ in the plotting of function, which is the basic difference between expressions and functions: functions already know what the independent variable is ($x$, or some other symbol), so they know that when we type $-5..5$ in the plot( ) command, we referring to $x$. Expressions don’t know this and need to be told.

To learn more about function plotting, let’s take another example: solve the inequality $x^2 < x$. Our technique is to treat each side of the inequality as a function, say $f(x) = x$, $g(x) = x^2$. If we plot both functions on one set of axes, then we can look for those $x$-values where the graph of $f(x)$ is higher than the graph of $g(x)$. Those $x$-values are the solutions.

```
> f:=x->x;
> g:=x->x^2;
> plot({f,g});
```
Not specify a domain and range, Maple used its default domain here. This is not very useful for our requirement.

> plot({f, g}, -2 .. 2);

This is much better than the previous one, here we specified the domain [-2, 2]. In the following, we go further to specify the range [-3, 3].

> plot({f, g}, -2 .. -3, -3 .. 3);

For which $x$-values does the graph of $f$ lie above that of $g$?

3. Exercises

- **1.** Find a solution to $x \sin(x) + 2 = 0$ a) between $\pi$ and $\frac{3\pi}{2}$; b) between $\frac{3\pi}{2}$ and $2 \pi$.

- **2.** Define $g(x) = \frac{x^2 + x - 2}{x^2 - 4x + 3}$ a) plot $g$ on [0, 5]; b) plot $g$ on [0.95, 1.05].

- **3.** Define $f(x) = \frac{\sqrt{3 + 2x^2}}{x + 5}$ a) find $f(100.0)$ to 10 decimal places; b) find $f(2000.0)$ to 10 decimal places.

- **4.** Use a graph to find all $x$ such that the inequality $5 x^3 < 8 x - 1$ is true.

- **5.** Define $f(x) = x^2$ and $g(x) = \sin(x)$
  a) Define $s(x)$ as the sum of $f(x)$ and $g(x)$, plot $f(x)$, $g(x)$ and $s(x)$ in the same window.
  b) Define $p(x)$ as the product of $f(x)$ and $g(x)$, plot $f(x)$, $g(x)$ and $p(x)$ in the same window.
  c) Define $c(x)$ as the composition of $f(g(x))$, plot $f(x)$, $g(x)$ and $c(x)$ in the same window.
  d) From the plots, can you see how $f(x)$, $g(x)$ are used to form each of $s(x)$, $p(x)$ and $c(x)$?

4. Supplementary Exercises

- **1.** Plot the graph of linear function $f(x) = ax + b$ when:
  1) $a = 1, b = 0$; 2) $a = 4, b = 0$; 3) $a = -1, b = 0$;
  4) $a = 0, b = 3$; 5) $a = -1, b = 2$; 6) $a = \frac{1}{2}, b = \frac{5}{7}$.
  Solve $f(x) = 0$ for $x$ in terms of $a$, $b$, then check the solutions on all the graphs.
• 2. Plot the graph of quadratic function \( f(x) = ax^2 + bx + c \) when:

1) \( a = 0, b = 4, c = 1 \); 2) \( a = 1, b = 0, c = 3 \);

3) \( a = 1, b = -2, c = 3 \); 4) \( a = \frac{1}{2}, b = \frac{3}{4}, c = 10 \).

Solve \( f(x) = 0 \) for \( x \) in terms of \( a, b, c \), then check the solutions on all the graphs.

• 3. Plot the graph of cubic polynomial function \( f(x) = ax^3 + bx^2 + cx + d \) when \( a = 1, b = 2, c = 3, d = 4 \). Solve the equation \( f(x) = 0 \) numerically from the graph. Can you solve \( ax^3 + bx^2 + cx + d = 0 \)?

• 4. Find the approximate range of \( f(x) = \frac{401}{11x^2-42x+48} \).

• 5. Define functions \( f(x) = \frac{\sqrt{x^2+3}}{2x-1} \) and \( g(x) = \frac{\sqrt{x+3}}{x-2} \), find a decimal value of 

  a) \( (f+g)(5) \); b) \( (f/g)(3) \); c) \( (fg)(4) \). (hint: a) \( f(5)+g(5) \); b) \( \frac{f(3)}{g(3)} \); c) \( f(4)g(4) \)

• 6. Consider the family of functions defined by \( y = Ax^2 + Bx + C \), for any values of \( A \) and \( B \), with nonzero \( A \).

  a) What can you say about each member of this family?
  b) Find the member of the family which passes through the points \((4,4)\) and \((5,2)\).
  c) Find the member which passes through the points \((4,4)\) and \((6,7)\).
  d) Plot the functions from parts b) and c) in the same window for \( x \) in \([-1, 7]\).

• 7. The relation \( 2y^2 + xy - 5 = 0 \) implicitly defines functions of the form \( y = y(x) \).

  a) Find the functions and plot their graphs in the same window.
  b) Plot these functions using the command **implicitplot**. (Look in the “plots” package.)

• 8. Plot the graph of \( f(x) \) and then the graph of \( |f(x)| \): a) \( f(x) = x \); b) \( f(x) = 2x + 1 \); c) \( f(x) = x^2 - 6x + 5 \); d) \( f(x) = \cos(x) \).
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9. For each $f(x)$, define $g(x)$ such that the graph of $g(x)$ is the graph of $f(x)$ shifted up by two units. Plot $f(x)$ and $g(x)$ in the same window. a) $f(x) = x^2$; b) $f(x) = \cos(x)$; c) $f(x) = \sin(x) + \frac{1}{x}$.

10. Repeat the previous problem, shifting to the right two units instead.

11. A function $f(x)$ is said to be even if $f(x) = f(-x)$ for each $x$ in the domain. A function $f(x)$ is said to be odd if $f(x) = -f(-x)$ for each $x$ in the domain. Decide whether each function is even, odd or neither, by plotting on an appropriate interval. a) $f(x) = x^2$; b) $f(x) = x^3$; c) $f(x) = x^2 + x^3$; d) $f(x) = 1 + x^2 - x^4$; e) $f(x) = \cos(x)$; f) $f(x) = \sin(x)$; g) $f(x) = \cos(x) + x^2$; h) $f(x) = \sin(x) - 2x + x^3$.

12. Plot $\sin(x)$ and $\sin(3x)$ in the same window for $x$ in the interval $[0, 2\pi]$. a) What can you say about the period and amplitude of each of these functions? b) If $n$ is a positive integer, how would the graphs of $\sin(x)$ and $\sin(nx)$ differ?

13. a) Plot $\sin(x)$ and $\sin(.1x^2)$ in the same window for $x$ in the interval $[-10, 10]$; b) Find the period and amplitude of each of these functions.

14. Plot each function (or pair) after you have predicted the appearance of the graph(s). a) $\sin(x)$, $x$ in $[0, 8]$; b) $\{ \sin(x), \sin(x + .5) \}$, $x$ in $[0, 2\pi]$; c) $\{ \sin(x), \sin(2x) \}$, $x$ in $[0, 2\pi]$; d) $\{ \sin(x), \frac{\sin(x)}{2} \}$, $x$ in $[0, 2\pi]$; e) $\frac{1}{10}x^2 + \sin(10x)$, $x$ in $[0, 2\pi]$; f) $\frac{1}{10}x^2 + \frac{1}{4}\sin(10x)$, $x$ in $[0, 8]$; g) $\sin(x) + \frac{\sin(15x)}{6}$, $x$ in $[0, 8]$. Which term causes the rapid oscillation along the smoother path? What causes the larger or smaller amplitude along the curve?

Lab 4. Powers and Logarithms

From section 1.4 in the textbook, we know powers and logarithms as numbers. However, sometimes it’s better to regard them as functions. How would these appear in Maple? This Lab is to help you understand more about powers and logarithms by exploiting the power and logarithm functions in Maple.

1. Power functions
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As we learned in Lab 1, Maple uses "^" to represent the power function. To write $a$ to the power $r$:

\[ a^r; \]
\[ a^0; \]

Maple treats $a$ (any number) to the power 0 as 1 directly, which is the axiom stated in Definition 1.4.1 in the textbook.

Next, we try to verify the Proposition 1.4.2 in text book.

\[ a^s*a^t; a^{(s+t)}; \]
\[ a^s*a^t-a^{(s+t)}; \]

In Maple, $a^s$ and $a^{(s+t)}$ are different symbols. That’s why when we ask Maple to compute the difference of them, it gives us nonzero answer. If we use the command `simplify( )`, Maple will give you the 0 answer.

\[ \text{simplify(%)}; \]

In fact, if we ask Maple to simplify $a^s a^t$, it gives $a^{(s+t)}$, i.e. $a^s a^t = a^{(s+t)}$.

\[ \text{simplify(a^s*a^t)}; \]

Same method works for $\frac{a^s}{a^t} = a^{(s-t)}$, but not for $(a b)^r = a^r b^r$, nor for $(\frac{a}{b})^r = \frac{a^r}{b^r}$.

\[ \text{simplify(\frac{a^s}{a^t})}; \]
\[ \text{simplify(a^s/a^t-a^r*(s-t))}; \]
\[ \text{simplify(((a*b)^r-a^r*b^r))}; \]
\[ \text{simplify((a/b)^r-a^r/b^r)}; \]

Maple doesn’t understand this rule? Yes, it really does. Try the following:

\[ \text{solve((a*b)^r=a^r*b^r)}; \]
\[ \text{solve((a/b)^r=a^r/b^r)}; \]

Look at the solution set, this is Maple’s way to tell you that the equation is true for all numbers. i.e. these two equations are identities. What about $a^s t = a^{(st)}$?

\[ \text{simplify((a^s)^t-a^r*(s+t))}; \]
\[ \text{solve(%)=0}; \]

Using `solve( )`, it also shows that $(a^s)^t = a^{(st)}$ is an identity in Maple. Notice that Maple can not simplify $(a b)^r - a^r b^r$, $(\frac{a}{b})^r - a^r b^{(-r)}$, and $(a^s)^t - a^{(st)}$ to 0, which may bring trouble in symbolic computation. This is why sometimes symbolic computation doesn’t give you very neat result. Maple is a mathematical software built by people. Only after you understand the mathematics well, can you use the Maple well.

To complete the verification of Proposition 1.4.2 by Maple, we need to check: for $1 < a$ and $t < s$, $a^t < a^s$. There may be some other methods to check this. Here we introduce the power function $f(x) = a^x$, where $1 < a$.

\[ a:=2; f:=x->a^x; \]
\[ t:=3; s:=5; \]
\[ f(t); f(s); \]

By varying the value of $a$, $t$, and $s$, we will always have $f(t) < f(s)$ as long as $1 < a$ and $t < s$. With the help of graph, this may be seen clearer.
> plot(f);
Later, we will learn that this is a monotonic increasing function, hence \( f(t) < f(s) \)
as long as \( t < s \).

2. Logarithmic Functions
Maple only displays natural logarithm, it changes any base logarithm to base-\( e \) logarithm. This can be done because of the assertion from Problem 1.4.3. in text book. The natural logarithm in Maple is \( \ln(x) \). The other base logarithm can be written as \( \log[a](x) \), which means base-\( a \) logarithm of \( x \).

> a:='a'; \log[a](x);
The power function \( a^x \) and its corresponding logarithm function \( \log[a](x) = \frac{\ln(x)}{\ln(a)} \) are related to each other as inverse functions. You can see this clearly from the following.

> solve(x=a^y,y);
> solve(x=log[a](y),y);
> simplify(%);

Now, again, there are some properties about the logarithm function stated in Theorem 1.4.5. in text book. How does Maple understand this? Since we have \( \log[a](x) = \frac{\ln(x)}{\ln(a)} \), we only need to verify Theorem 1.4.5. for the natural logarithms. To enter \( e^x \) into Maple, we write \( \exp(x) \).

> simplify(exp(ln(x)));
> simplify(ln(exp(x)));
> solve(ln(exp(x))=x,x);
> ln(x*y)-(ln(x)+ln(y));
> simplify(%);
> solve(%,0);
> ln(x/y)-(ln(x)-ln(y));
> simplify(%);
> solve(%,0);
> ln(x^r)-r*ln(x);
> simplify(%);
> solve(%,0);
> ln(1);

Similarly to powers, the command \( \text{simplify()} \) in Maple can not recognize the identities: \( \ln(x y) = \ln(x) + \ln(y) \), \( \ln(\frac{x}{y}) = \ln(x) - \ln(y) \), \( \ln(x^r) = r \ln(x) \), etc. Thus it can’t really simplify the symbolic result as you expected. Sometimes, you have to do the simplification by hands. Therefore, it’s important for you to understand the theory before you start to use Maple. Like any other software, Maple is not perfect, some things can be done and some things can not be done by Maple.

3. Exponentials and Logarithms
Now, let’s plot \( y = e^x \), and \( y = x^n \) on the same set of axes, for various values of \( n \):
Appendix 2: Introduction to Maple V Labs

> plot({exp(x), x, x^2, x^3, x^4}, x=0..1);
Which curve is which? Hint: What is the $y$-intercept of $e^x$? of $x^n$? Find more details in Exercise 3.

Now plot the natural logarithm $\ln(x)$ together with $x^{(\frac{1}{n})}$ on the same set of axes, beginning with the domain $x = 0.50$. (If there are problems, try plotting with the domain $x = 1.50$ instead. The point $x = 0$ might cause minor trouble because $\ln(x)$ is not defined at $x = 0$.)

> plot({ln(x), x^(1/2), x^(1/3), x^(1/4), x^(1/5)}, x=0..50);
Which curve is which? Find more details in Exercise 4.

4. Exercises

• 1. There is another way to verify Proposition 1.4.2. in text book, i.e. choose specific numbers for $\alpha$, $\beta$, $r_1$, and $r_2$: 

> alpha:=2; beta:=3; r1:=1.34; r2:=3.4;
> alpha^r1*alpha^r2; alpha^(r1+r2);

By varying the values of $\alpha$, $\beta$, $r_1$, and $r_2$, you will find that $\alpha^{r_1} \alpha^{r_2} = \alpha^{(r_1+r_2)}$ is always true, as long as $\alpha$ is positive, $r_1$ and $r_2$ are rational numbers.

Use this method to check the other items in Proposition 1.4.2.

• 2. Similarly to the previous question, we can also check Theorem 1.4.5. in text book.

> alpha:=4; r1:=1.4; r2:=5.6; r:=-1;
> log[alpha](alpha^r); r;
> alpha^(log[alpha](r1)); r1;

By varying the values of $\alpha$, $r_1$, $r_2$, $r$, you will get $\log[\alpha](\alpha^r) = r$ and $\alpha^{(\log[\alpha](r1))} = r1$.

Use this method to check the other items in Theorem 1.4.5.

• 3. Repeat the first plot command in section 3 of this Lab. Each time, increase the upper limit of the domain by one (so plot using $x = 0..2$, then plot again with $x = 0.3$, and so on). As you increase the domain, you will eventually find a point where $e^x = x^3$ and beyond that point $x^3 < e^x$. Find the $(x, y)$ coordinates of this point. Continue to incrementally increase the domain, stop when you are able to display the point where $e^x = x^4$ and find the $(x, y)$ coordinates of this point as well. Give a rough sketch of this last graph, labeling the curves.
4. This is for the second plot command in section 3 of this Lab. Now increase the domain incrementally until you reach the point where the curve of \( x^{\frac{1}{3}} \) overtakes that of \( \ln(x) \). What are the coordinates of this point? What are the coordinates of the point where the curve of \( x^{\frac{1}{4}} \) overtakes that of \( \ln(x) \)? Where the curve of \( x^{\frac{1}{5}} \) overtakes that of \( \ln(x) \)? [Hint: Don’t increase the domain by only small amounts. You will need to use truly enormous values for the upper limit of the domain. Also, don’t bother to continue to plot \( x^{\frac{1}{3}} \) and \( x^{\frac{1}{4}} \) when looking for \( x^{\frac{1}{5}} \) to overtake \( \ln(x) \), as they will only get in the way.]

**Lab 5. Summations and Inductive Verification**

Clarity of expression depends on choice of words. In mathematics, it is choice of notation. In the first part of this Lab, we explore summation notation and its implementation in Maple. Traditionally, the validity of an indefinite summation formula is established by mathematical induction. Mathematical induction is one of the most fundamental techniques of proof in mathematics.

1. Summations: Finite and Infinite

The sum \( a_0 + a_1 + a_2 + ... + a_n \) is written, using the Greek \( \Sigma \) (sigma), as \( \sum_{k=0}^{n} a_k \), where \( k \) is the index, 0 is the initial value of \( k \), and \( n \) is the final value of \( k \). We can also express the sum of a function evaluated at consecutive points by \( \sum_{k=0}^{n} f(k) = f(1) + f(2) + f(3) + ... + f(n-1) + f(n) \). We may vary the starting and ending values of the index, but we always choose integers (or integer-valued variables). For example, \( \sum_{k=4}^{7} (3k^2 + 1) = (3\,4^2 + 1) + (3\,5^2 + 1) + (3\,6^2 + 1) + (3\,7^2 + 1) = 382 \).

One way to compute the sum in Maple is to use a `for ... from ... to ... do ... od;` construct that mimics the process of adding the terms:

```maple
> total:=0;
> for r from 4 to 7
> do
>   total:=total+(3*r^2+1);
> od;
> r:='r';
```

A more elegant and useful notation is

```maple
> sum(3*j^2+1, j=4..7);
```

In \( \Sigma \) notation variables can appear as limits \( \sum_{k=0}^{n} 2^k = 1 + 2 + 4 + 8 + ... + 2^{(n-1)} + 2^n \). In Maple, this is simply

```maple
> sum(2^k,k=0..n);
```
Example 1. Write $\sum_{k=71}^{123} (k^2+1)$ as a Maple sum statement and write \texttt{sum}((1/3)^k, k=0..n) in sigma notation. In each case, execute the sum in Maple.

Answer: the Maple sum statement for the first summation: \texttt{sum}(k^2+1, k=71..123);
the sigma notation for the second summation: $\sum_{k=0}^{n} (\frac{1}{3})^k$; hit the \texttt{ENTER} key by the end of the following two Maple commands to execute the sums in Maple.

\begin{verbatim}
> sum(k^2+1, k=71..123);
> sum((1/3)^k, k=0..n);
\end{verbatim}

Maple can return closed form expressions for many indefinite summations:

\begin{verbatim}
> sum(k^2, k=1..n);
> simplify(%);
> factor(%);
> sum(4*k^3-2*k^2, k=1..n);
> simplify(%);
> factor(%);
> sum(k^2, k=n..m);
> simplify(%);
> factor(%);
\end{verbatim}

Maple can also handle sums that add an infinite number of terms:

\begin{verbatim}
> sum((1/2)^n, n=1..infinity);
\end{verbatim}

However, sometimes the infinite sum makes no sense, as in adding 1s forever with:

\begin{verbatim}
> sum(1, k=0..infinity);
\end{verbatim}

Also, not every summation makes sense:

\begin{verbatim}
> sum((-1)^n/2, n=1..infinity);
\end{verbatim}

And Maple sometimes deals in half-truths:

\begin{verbatim}
> sum(x^n, n=0..infinity);
\end{verbatim}

which is fine when $x = \frac{1}{2}$ but not for $x = 2$. The moral is that infinity must be handled with caution and discretion.

2. Inductive verification
The formal statement of mathematical induction is as follows:
Suppose that $P(n)$ is a statement about the positive integer $n$. If it can be shown that

- the validity of the statement $P(k)$ implies the validity of the statement $P(k+1)$, and that
- $P(1)$ is valid,
then it follows that $P(n)$ is true of all positive integers.
In this section, we will:

- Use Maple’s sum command to supply a formula.

- Use Maple to carry out the algebraic computations necessary to establish that the validity of $P(k)$ implies the validity of $P(k + 1)$.

- Verify that $P(1)$ is indeed true.

The following is a record of a Maple session. In this session the “inert” command $\texttt{Sum}$ and the $\rightarrow$ are used to construct mathematical equations as propositions about $n$. The commands $\texttt{rhs}$ and $\texttt{lhs}$ are used to separate the right- and left-hand sides of equations. The command $\texttt{value}$ is used to evaluate inert sums.

```
> with(student):
We use the $\texttt{Sum}$ command to obtain sigma notation statements:
> Sum(i^3, i=1..3);
The $\texttt{sum}$ command gives values and formulas:
> sum(i^3, i=1..3);
Correct! $1 + 8 + 27 = 36.$
> sum(i^3, i=1..n);
> normal(%);
> factor(%);
It is never clear beforehand which form of an expression will be most useful.
Now construct the proposition $P(n)$ for an inductive proof with:
> P:=n->Sum(i^3, i=1..n)=sum(i^3, i=1..n);
> P(3);
> P(n);
> normal(%);
> factor(%);
Now we proceed with an inductive proof of $P(n)$:
Suppose that for some positive integer $k$, $P(k)$ is true. Specifically, assume the truth of $P(k)$ for one particular $k$:
> P(k);
Adding $(k + 1)^3$ to both sides preserves equality, so:
> %+((k+1)^3=(k+1)^3);
```
By inspection, the left-hand side of this equation is the sum $1^3 + 2^3 + \ldots + (k + 1)^3$. If the right-hand side can be shown to be equal to rhs(P(k+1)), we have established that the validity of P(k) implies the validity of P(k+1).

Select the right-hand side of our last equation for comparison to rhs(P(k+1)):

```maple
> factor(rhs(%));
```

and compare to the factored form of rhs(P(k+1)):

```maple
> rhs(P(k+1));
> factor(%);
```

They match! To complete the proof we need only determine if P(1) is true.

```maple
> P(1);
```

And now, for the truly lazy:

```maple
> value(%);
```

3. Exercises

We will investigate the properties of several forms of summation. In each problem from 1 to 4, after you type the expression, but before you press the **ENTER** key, ask yourself what answer to expect.

- 1. We begin with several simple sums. A sum’s terms can be constant. Write each of the following sums in Σ notation and evaluate:
  ```maple
  sum(3, k=0..12)
  sum(-2, k=-3..3)
  ```

- 2. Investigate the sequence of summations that begins
  ```maple
  sum(1, m=1..n)
  sum(m, m=1..n)
  sum(m^2, m=1..n)
  sum(m^3, m=1..n)
  ```
  Examine each summation in both normal and factored form. List several patterns that you see.

- 3. Investigate the sequence of summations that begins
  ```maple
  sum(1/(k*(k-1)), k=2..n)
  sum(1/(k*(k-1)*(k-2)), k=3..n)
  sum(1/(k*(k-1)*(k-2)*(k-3)), k=4..n)
  ```
As before, examine each summation in both normal and factored form and list several patterns that you see.

4. Define the polynomial $p$ by

\[ P := x \rightarrow x^3 + 3x^2 + 2x - 5: \]

Use the sum command to evaluate $\sum_{k=3}^{21} p(2k + 1)$ and $\sum_{i=0}^{3n-1} p(1 + \frac{i}{n})$. We can evaluate $p$ taken at elements of a list:

\[ x := [0, 0.2, 0.4, 0.6, 0.8, 1.0]: \]

Evaluate the sum:

\[ \text{sum}(p(x[i]), i=1..6); \]

Longer lists may be generated with the sequence function

\[ \text{pts} := [\text{seq}(2+(3*k/20), k=0..20)]; \]

Now calculate

\[ \text{sum}(p(\text{pts}[j])*(3/20), j=1..20); \]

This list calculation is a special form of a Riemann sum. This special form of a Riemann sum, called a "left-sum", has the type

\[ R = f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2) + \ldots + f(x_{n-1})(x_n - x_{n-1}) \]

where $x_1 < x_2 < \ldots < x_n$. In Maple, we can write this sum using two statements. For $f(x) = \sin(x)$, determine $R$ when $x_1 = 0$, $x_2 = \frac{\pi}{20}$, $x_3 = \frac{\pi}{10}$, $x_4 = \frac{3\pi}{20}$, $x_5 = \frac{\pi}{5}$, and $x_6 = \frac{\pi}{4}$ with

\[ f := x \rightarrow \sin(x); \]

\[ x := [\text{seq}(\pi/4*(k/5), k=0..5)]; \]

\[ R := \text{sum}(f(x[i])*(x[i+1]-x[i]), i=1..5); \]

Consider a Riemann sum for which $x_i = \frac{i}{n}$ so that $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, ..., and $x_n = 1$. Then $R = \sum_{i=1}^{n} \frac{f(\frac{i}{n})}{n}$. Define $f$ by $x \rightarrow x^2$. Enter $R$ as a Maple \texttt{sum} statement and use

\[ \text{limit}(R, n=\text{infinity}); \]

to find the infinite summation of $R$.

5. Use Maple to produce and verify closed forms for the indefinite summations:

\[ \sum_{i=1}^{n} i^4, \sum_{i=1}^{n} i^5, \sum_{i=1}^{n} \frac{1}{i(i+1)}, \sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)}, \sum_{i=1}^{n} x^i. \]
6. A trigonometric identity that is fundamental to the study of Fourier series is
\[
\frac{1}{2} + \left( \sum_{j=1}^{n} \cos(jx) \right) = \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{x}{2})}
\]
Prove this identity with induction:
1. Define \( P(n) \).
2. Prove that the validity of \( P(k + 1) \) follows from assuming \( P(k) \) is true:
   a. Consider \( \text{rhs}(P(k)) + \cos((k+1)*x) = \text{rhs}(P(k+1)) \). \textit{Why is this expression correct?}
   b. Multiply both sides by the denominator to clear fractions.
   c. Use \texttt{expand(\%,\textit{sin,cos})} to multiply the terms without applying trigonometric sum and product formulas.
   d. Use \texttt{combine(\%,\textit{trig})} to reduce the expression, this time applying the trigonometric sum and product formulas.
3. Establish that \( P(0) \) is true.

\[
\text{Lab 6. What is a Sequence}
\]
From the textbook, we learned the math definition of a sequence and a subsequence. What is the Maple’s definition of a sequence? How would Maple display a sequence?

1. Maple’s Definition of Sequence
Informally, a sequence is an infinite list of real numbers, written down one after the other. That is, for every positive integer \( n \), we are given a real number, which we write as \( a_n \). The precise definition of a sequence is that a sequence \( \{a_n\} \) is a function \( a: \mathbb{N} \rightarrow \mathbb{R} \) where \( a_n := a(n) \) for \( n=1, 2, 3, ..., \) i.e. the value \( a(n) \) is then written as \( a_n \).
In Maple, we define a sequence \( a_n = \frac{n}{2n+1} \) as:
\[
> a:=n->n/(2*n+1);
\]
and list the terms of the sequence \( \{a_n\} \) by using a loop:
\[
> \text{for } i \text{ from } 1 \text{ by } 1 \text{ to } 100 \text{ do }
> a(i)
> \text{od;}
\]
or we can use \texttt{print(i,a(i))} instead of \texttt{a(i)}:
\[
> \text{for } i \text{ from } 1 \text{ by } 1 \text{ to } 100 \text{ do }
> \text{print(i,a(i))}
> \text{od;}
\]
Also we can use \texttt{seq( )} to create a sequence.
Now, with this, it’s not hard for us to display the example sequences shown in the textbook.

i) \( a_n = c \)
   
   ```
   > a:=n->c;
   > for i from 1 by 1 to 10 do
   > print(i,a(i))
   > od;
   > seq(c,i=1..10);
   ```

ii) \( a_n = \frac{1}{n} \)
    
    ```
    > a:=n->1/n;
    > for i from 1 by 1 to 10 do
    > print(i,a(i))
    > od;
    > seq(1/n,n=1..10);
    ```

iii) \( a_n = (-1)^n \)
     
     ```
     > a:=n->(-1)^n;
     > for i from 1 by 1 to 10 do
     > print(i,a(i))
     > od;
     > seq((-1)^n,n=1..10);
     ```

iv) \( a_n = n(\sqrt{n} - 1) \)
    
    ```
    > a:=n->n*(sqrt(n)-1);
    > for i from 1 by 1 to 10 do
    > print(i,a(i))
    > od;
    > seq(n*(sqrt(n)-1), n=1..10);
    ```

v) \( a_n = \frac{n+1}{n^2-3n+2} \)
    
    ```
    > a:=n->(n+1)/(n^2-3*n+2);
    > for i from 3 by 1 to 20 do
    > print(i,a(i))
    > od;
    > seq((n+1)/(n^2-3*n+2), n=3..20);
    ```

vi) \( a_n = \sqrt{n^2+n} - \sqrt{n^2-n} \)
    
    ```
    > a:=n->sqrt(n^2+n)-sqrt(n^2-n);
    > for i from 1 by 1 to 10 do
    > print(i,a(i))
    > od;
    > seq(sqrt(n^2+n)-sqrt(n^2-n),n=1..10);
    ```
vi) \( a_n = \sqrt{n^2 + 1} - \sqrt{n^2 - 1} \)

\[
\begin{align*}
&\text{a}:=n->\sqrt{n^2+1}-\sqrt{n^2-1}; \\
&\text{for i from 100 by 1 to 110 do} \\
&\quad \text{print}(i,a(i)); \\
&\quad \text{evalf}(a(i)); \\
&\quad \text{od}; \\
&\text{seq}(\sqrt{n^2+1}-\sqrt{n^2-1},n=100..110);
\end{align*}
\]

vii) \( a_n = \frac{(-1)^n}{n} \)

\[
\begin{align*}
&\text{a}:=n->(-1)^n/n; \\
&\text{for i from 1 by 1 to 10 do} \\
&\quad \text{print}(i,a(i)) \\
&\quad \text{od}; \\
&\text{seq}((-1)^n/n,n=1..10);
\end{align*}
\]

viii) \( a_n = (1 + \frac{1}{n})^n \)

\[
\begin{align*}
&\text{a}:=n->(1+1/n)^n; \\
&\text{for i from 1000 by 1 to 1020 do} \\
&\quad \text{print}(i,\text{evalf}(a(i))) \\
&\quad \text{od}; \\
&\text{seq}(\text{evalf}((1+1/n)^n,10),n=1000..1020);
\end{align*}
\]

The Fibonacci’s sequence is defined by a recurrence formula: \( u_1 = 1, u_2 = 1, u_n = u_{n-1} + u_{n-2}. \)

\[
\begin{align*}
&u[1]:=1; \\
&u[2]:=1; \\
&\text{for i from 3 by 1 to 10 do} \\
&\quad u[i]:=u[i-1]+u[i-2]; \\
&\quad \text{od}; \\
&\text{seq}(u[i],i=1..10);
\end{align*}
\]

2. Maple’s definition of Subsequence

To define a subsequence, similarly as the definition of sequence, if we have \( a_n \), we can define the subsequence \( b_k = a_{2k} \) or \( b_k = a_{2k+1}. \)

\[
\begin{align*}
&\text{a}:=n->(-1)^n; \\
&\text{b}:=k->a(2*k); \\
&\text{for i from 1 by 1 to 10 do} \\
&\quad \text{print}(i,b(i)) \\
&\quad \text{od}; \\
&\text{a}:=n->n!/n^n; \\
&\text{b}:=k->a(2*k+1); \\
&\text{for i from 1 by 1 to 10 do} \\
&\quad \text{print}(i,b(i))
\end{align*}
\]
Exercises

1. Use Maple’s definition of a sequence to display several items of the following sequences and, at the same time, to verify which of the following sequences are monotonic and bounded.
   (a) \( \{a_n\} \), where \( a_n = \frac{n!}{n^n} \) for \( n \in \mathbb{N} \);
   (b) \( \{b_n\} \), where \( b_n = \frac{n!}{2^n} \) for \( n \in \mathbb{N} \);
   (c) \( \{c_n\} \), where \( c_n = \frac{n^2 + 2n + 1}{n^2 - 3} \) for \( n \in \mathbb{N} \);
   (d) \( \{d_n\} \), where \( d_n = \frac{2^n}{n!} \) for \( n \in \mathbb{N} \);
   (e) \( \{e_n\} \), where \( e_n = n((-1)^n) \) for \( n \in \mathbb{N} \).

2. For the following 10 sequences, write the first 100 terms of the sequence.
   1) \( a_n = \frac{n}{2n+1} \); 2) \( a_n = \frac{4n-3}{3n+4} \); 3) \( a_n = (-1)^{(n-1)} n \); 4) \( a_n = (-\frac{2}{3})^n \); 5) \( a_n = (1.3.5...(2n-1))/n! \); 6) \( \{(-1)^{(n+1)}n!\} \); 7) \( a_n = \{\sin(\frac{1+n}{2})\} \); 8) \( a_n = \cos(n\pi) \); 9) \( a_1 = 1, a_{n+1} = \frac{1}{1+a_n} \); 10) \( a_1 = 0, a_2 = 1, a_n = a_{n-1} + a_{n-2} \).

3. For the following exercises, find a formula for the general term \( a_n \) of the given sequence assuming that the pattern of the first few terms continues.
   1) \( \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \} \); 2) \( \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \} \); 3) \( \{1, 4, 7, 10, \ldots \} \); 4) \( \{\frac{5}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{39}, \ldots \} \); 5) \( \{-1, 2, -6, 24, \ldots \} \); 6) \( \{\frac{3}{2}, -\frac{9}{4}, \frac{27}{8}, -\frac{81}{16}, \ldots \} \); 7) \( \{\frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{3}{9}, \ldots \} \); 8) \( \{0, 2, 0, 2, 0, 2, \ldots \} \).

Lab 7. Convergence of Sequences and the Notion of a Limit
How to find out using Maple that a sequence is convergent or divergent? What is the limit if a sequence is convergent? In this Lab you will get all the answers.

1. Evaluation of Limits in Maple
Let us recall the mathematical definition of the convergence of a sequence: let \( \{a_n\} \) be a sequence and \( L \) a real number, we say that \( \{a_n\} \) converges to \( L \), or that \( L \) is the limit of \( \{a_n\} \), i.e. \( \lim_{n \to \infty} a_n = L \), if for every \( \varepsilon > 0 \) there exists an integer \( N \) such
that $|a_n - L| < \varepsilon$ for all $n$ greater than or equal to $N$. If a sequence $\{a_n\}$ does not converge to any real number, then we say that $\{a_n\}$ diverges. There are also similar definitions for $\lim_{n \to \infty} a_n = \infty$ and $\lim_{n \to \infty} a_n = -\infty$.

Maple can not do the proof and the analysis. We need to learn all the techniques to prove or to find out if a given sequence is convergent or divergent. However, Maple may help you to verify the convergence or divergence of a sequence. Sometime, it may give you some idea of the given sequence if you use Maple properly. We can use the Maple command \texttt{limit(a(n),n=infinity);} to evaluate the limit of the sequence $\{a_n\}$.

For example, to find out the limit of $a_n = \frac{(n^2)^{\frac{1}{3}} \sin(n!)}{n+1}$:

\begin{verbatim}
> a:=n->root(n^2,3)*sin(n!)/(n+1);
> limit(a(n),n=infinity);
\end{verbatim}

This means that the given sequence $\{a_n\}$ is convergent, and the limit is 0. However, we need to prove the result by ourselves, by hands. Can you prove that $\lim_{n \to \infty} \frac{n(\frac{2}{3})\sin(n!)}{n+1} = 0$?

\begin{verbatim}
> a:=n->((-2)^n+3^n)/((-2)^(n+1)+3^(n+1));
> limit(a(n),n=infinity);
\end{verbatim}

For this case, Maple does not help. However, if we divide the numerator and the denominator of $a_n$ by $3^n$ first, then Maple is able to evaluate the limit!

\begin{verbatim}
> a:=n->((-2/3)^n+1)/((-2/3)^n*(-2)+3);
> limit(a(n),n=infinity);
\end{verbatim}

This is a skill that used very often. Let’s see some other examples.

\begin{verbatim}
> a:=n->ln(n)/n;
> limit(a(n),n=infinity);
> a:=n->2^n/n!;
> limit(a(n),n=infinity);
> a:=n->n!/n^n;
> limit(a(n),n=infinity);
\end{verbatim}

We want to find the limit of $a_n = 1*3*5*...*(2^n-1)/((2^n)n-1)$. The hard point here is how to express the numerator in Maple. In fact $1*3*5*...*(2^n-1) = \frac{1(2n)!}{2^n n!}$.

\begin{verbatim}
> a:=n->(2^n)!/(2^n*n!*(2*2^n));
> limit(a(n),n=infinity);
\end{verbatim}

2. Exercises

- 1. Find the limits of the following sequences:
  
  a) $a_n = \frac{n \sin(3n+1)}{n^2+1}$;  
  b) $b_n = \frac{(\sum_{i=1}^{n} i) \cos(n!)}{n^3+1}$.
2. Use Maple to evaluate the limits of the following sequences, while do the proof by yourself using the properties of convergent sequences.

\[ a_n = \sum_{i=1}^{n} \frac{1}{i}; \quad b_n = \frac{(n+2)}{n}; \quad c_n = \sqrt{n^2 + n} - \sqrt{n^2 - n}; \quad d_n = \sqrt{n^4 + n^2} - \sqrt{n^4 - n}; \quad e_n = \frac{\sqrt{n^2 + 2n} - n}{\sqrt{n^2 + 2n} - n}. \]

3. Draw the graph of the sequence \( \{ \frac{(-1)^n (n+1)}{n} \} \). Is the sequence convergent or divergent?

4. For the following sequences, determine whether the sequence converges or diverges. If it converges, find the limit.

1) \( a_n = \frac{1}{4n}; \) 2) \( a_n = 4 \sqrt{n}; \) 3) \( a_n = n^2 - 1; \) 4) \( a_n = \frac{4n^2 - 3}{3n^2 + 1}; \) 5) \( a_n = \frac{n^2}{n^2 + 1}; \) 6) \( a_n = \frac{1}{n^2 + n!}; \) 7) \( a_n = \frac{1}{n^2 + n!}; \) 8) \( a_n = \frac{(-1)^{n-1}}{n^2 + 1}; \) 9) \( a_n = \frac{1}{n^2 + 1}; \) 10) \( a_n = 2 + \frac{21}{n}; \) 11) \( a_n = \cos \left( \frac{n \pi}{2} \right); \) 12) \( a_n = \sin \left( \frac{n \pi}{2} \right); \) 13) \( a_n = \frac{n}{n}; \) 14) \( a_n = \arctan \left( \frac{2n}{n^2 + 1} \right); \) 15) \( a_n = \arctan \left( \frac{2n}{n^2 + 1} \right); \) 16) \( a_n = \sin(n); \) 17) \( a_n = \frac{3 + (-1)^n}{n^2}; \) 18) \( a_n = \frac{n!}{n^2 + 1}; \) 19) \( a_n = \frac{\ln(n^2)}{n}; \) 20) \( a_n = (-1)^n \sin \left( \frac{1}{n} \right); \) 21) \( a_n = \sqrt{n^2 + 2 - \sqrt{n}}; \) 22) \( a_n = \frac{1}{3n^2}; \) 23) \( a_n = \frac{n^2}{n^2 + 1}; \) 24) \( a_n = \frac{n^2}{n^2 + 1}; \) 25) \( a_n = \ln(n+1) - \ln(n); \) 26) \( a_n = \ln \left( \frac{1}{2} \right); \) 27) \( a_n = \frac{\cos(n^2)}{2n}; \) 28) \( a_n = \frac{\cos(n)}{n^2 + 1}; \) 29) \( a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{1}{n^2}; \) 30) \( a_n = \frac{1}{n^2} \) and \( n^2 + 1; \) 31) \( a_n = \frac{n!}{n^2}; \) 32) \( a_n = \frac{(-3)^n}{n!}; \) 33) \( a_n = \frac{n^3}{n^3}; \) 34) \( a_n = \frac{3n^5 + 5n^5}{n^5} \).

5. For the following sequences, use Maple to list some terms to see if the given sequence is increasing, decreasing, or not monotonic. Then try to prove the assertions by yourself.

\[ a_n = \frac{1}{3n^2 + 1}; \] 2) \( a_n = \frac{1}{5n}; \) 3) \( a_n = \frac{n^2}{n^2 + 1}; \) 4) \( a_n = \frac{3n^4 + 4}{2n^5}; \) 5) \( a_n = \cos \left( \frac{1}{2n} \right); \) 6) \( a_n = 3 + \frac{(-1)^n}{n}; \) 7) \( a_n = \frac{1}{n^3 + n^3}; \) 8) \( a_n = \frac{1}{5n^3 + 1}. \]

Lab 8. Applications of Limits: The Number e

In this Lab we investigate different ways in which the natural base \( e \) arises. At the same time, we will also explore some functions related to the natural base \( e \).

1. Definitions of the Number \( e \)

You are probably well familiar with the irrational constant \( \pi \) which occurs throughout trigonometry and geometry. It is defined in an easily understood way: it’s just the
ratio of the circumference of a circle to its diameter. In algebra and calculus, another irrational constant often appears, the so-called natural base, denoted by \( e \). Like \( \pi \), the decimal expansion of \( e \) requires an infinite number of decimal places. In this section, we will explore several different (but inter-related) ways in which this number arises, and in the process determine its approximate value.

First, we consider the following sequence \( a_n = (1 + \frac{1}{n})^n \) In the textbook, it is shown that \( \{ a_n \} \) is an increasing and bounded sequence. From this, we have the first definition of the number \( e \):

\[
e = \lim_{n \to \infty} (1 + \frac{1}{n})^n
\]

Compute \( a_n \) for several values of \( n \) including \( n=10, n=100, n=1000, \) and \( n=10^6 \). What value does \( a_n \) appear to approach as \( n \) gets very large?

```maple
> a:=n->(1+1/n)^n;
> evalf(a(10),10); evalf(a(100),10); evalf(a(1000),10);
> evalf(a(10e+6),10);
```

We can check the value of the number \( e \) by calling the exponential function from Maple. Remember to specify the number of decimal places you want it to display.

```maple
> evalf(exp(1),16);
```

To get this approximation, we need to take \( n = 10^{15} \) in the sequence of \( \{ a_n \} \).

```maple
> evalf(a(10e+15),17);
```

Next, by the squeeze property and the proof shown in the textbook, we also have another equivalent definition for the number \( e \), namely \( e = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!} \) which can be also written as \( e = \sum_{k=0}^{\infty} \frac{1}{k!} \).

Evaluate several terms to approximate the value of the number \( e \).

```maple
> b:=n->sum(1/k!,k=0..n);
> evalf(b(10),17); evalf(b(17),17);
```

This sequence \( b_n = \sum_{k=0}^{n} \frac{1}{k!} \) converges very fast. We only need to take \( n=17 \) to get the approximation \( e = 2.718281828459045 \).

**2. Exponential Function and Logarithmic Function**

The first definition can be generalized, allowing us to define a whole function, i.e. the exponential function:

\[
e^x = \exp(x) = \lim_{n \to \infty} (1 + \frac{x}{n})^n
\]

Maple knows about this function. It calls it \( \exp( ) \). \( \exp(1) \) should give you the value of the number \( e \).

```maple
> exp(1); evalf(%,20);
```

Let’s plot the graph of this function to see how it looks like.

```maple
> plot(exp(x),x=-2..2);
```

In addition, there is an inverse function to \( \exp(x) \), called the natural logarithm and denoted by \( \ln(x) \). Maple knows about it as well and calls it \( \ln( ) \) or \( \log( ) \).

```maple
> ln(1);
```
> evalf(ln(17.5));
> ln(exp(1));
> plot(exp(x), x=-2..5);

Because ln(x) and exp(x) are inverse functions for each other, they obey cancelation equations which read:

\[
\ln(\exp(x)) = x \\
\exp(\ln(x)) = x
\]

> ln(exp(10));
> exp(ln(a));

3. Exercises

• 1. Solve each of the following equations by plotting both the function on the left-hand side and the function on the right-hand side on a single set of axes and finding all points where the two graphs intersect. Copy down a plot in each case, and indicate the points of intersection.

a) \( e^x = x^3 \) Adjust the domain of the plot to ensure you find all solutions. Also, be sure you know which graph is which (Hint: these functions behave differently at \( x=0 \))

b) \( e^{(-x)} = -x^2 + 3x \)

c) \( x = 3 \ln(x) \)

d) \( e^x = e^{(-x)} \)

e) \( x^4 - 5x^2 + 1 = e^{(-x)} \)

The roots in parts (a) and (c) should have been the same. Explain why, using the cancelation equations.

• 2. Find the limit of \( \lim_{n \to \infty} a^{(\frac{1}{n})} \) in Maple, assuming \( a > 0 \).

• 3. Find the limit of \( \lim_{n \to \infty} (1 - \frac{1}{n^2})^n \) in Maple, and prove the result by yourself using the Bernoulli’s inequality.

• 4. In Maple, evaluate the limit when \( n \) approaches infinity for the following sequences

a) \( a_n = (1 - \frac{1}{n})^n \);

b) \( a_n = (1 + \frac{m}{n})^n \);
c) \(a_n = (1 - \frac{m}{n})^n\);

d) \(a_n = (1 + \frac{1}{n^2})^n\).

Try to compute these limits by yourself.

**Lab 9. Study of Functions**

First we will recall the differences between expressions and functions in Maple that were discussed in previous Labs. Using the function notation, we will examine inequalities, domain of functions, range of functions, composition of functions, piecewise-defined functions, emphasizing graphical techniques.

1. Functions and Expressions, Plotting of Functions and Solving Inequalities

Define the expression \(e_1 = x^3 + 3x\). To evaluate it at \(x = 1/2\), we use the \texttt{subs()} command (short for 'substitute'). To get a decimal answer, we must use \texttt{evalf()}.

```maple
> e1:=x^3+3*x;
> subs(x=1/2,e1);
> evalf(%);
```

Maple defines functions by using the arrow notation. To make it, you first type a dash – and then follow it with a 'greater than' symbol > to get ->. Define the following function:

```maple
> f:=x->x^3+3*x;
```

This notation says that \(f\) is the function that takes \(x\) as input, computes \(x^3 + 3x\), and returns the result as output. To evaluate a function, we do not need the \texttt{subs()} command. Type:

```maple
> f(1/2);
```

The plotting commands for functions are slightly different than those for expressions. We practice them by using plots to help solve the inequality \(x^2 < x\). By this, we mean finding all \(x\)-values that are greater than their own squares. Our technique is to treat each side of the inequality as a function, say \(f(x) = x\) and \(g(x) = x^2\). If we plot both functions on one set of axes, then we can look for those \(x\)-values where the graph of \(f(x)\) is higher than the graph of \(g(x)\). Those \(x\)-values are the solutions.

```maple
> f:=x->x;
> g:=x->x^2;
> plot({f,g});
```

Since we did not specify a domain and range, Maple used its default domain here. The resulting plot is not very useful.

```maple
> plot({f,g},-2..2);
```

Now, that's better. Here we have specified the domain \(x\) in \([-2,2]\).
```maple
plot({f,g},-2..2,-3..3);
```

Even better. This time we’ve also specified the range \( y \) in \([-3,3]\). For which \( x \)-values does the graph of \( f \) lie above that of \( g \)?

Notice the command syntax: We simply said \(-2.2\) instead of \( x = -2.2 \) when specifying the domain. That’s the basic difference between functions and expressions: functions already know that when we type \(-2.2\) in the \texttt{plot()} command, we are referring to \( x \). Expressions don’t know this and need to be told.

2. Domain, Range of Functions and Composition of Functions

As it is not always clear what exactly is the domain of a function given by a formula, it is assumed that the domain in such a case in the largest set of real numbers for which the formula makes sense. For example, the function \( f(x) = \frac{1}{x^2-1} \) has the domain \( \text{Dom}(f) = (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \), where the symbol ‘\( \cup \)’ denotes the union.

```maple
f:=x->1/(x^2-1);plot(f,-2..2,-10..10);
```

If we plot out the graph for this function, we will find that there are lines at \( x = -1 \) and \( x = 1 \), which are exactly the undefined points. This is how Maple deals with the discontinuity (details in the coming up labs). Also from the graph we know that the range of this function \( \text{Range}(f) = (-\infty, 0) \cup (0, \infty) \), i.e. the value of this function could be any real number but zero.

```maple
f:=x->sqrt(x^2-1);
plot(f); plot(f,-1..1); plot(f,-1.1..1.1); f(-1); f(1); f(0);
```

From this, \( \text{Dom}(f) = (-\infty, -1] \cup [1, \infty) \), and \( \text{Range}(f) = [0, \infty) \).

There is a way, called composition, of combining two functions or several functions to define a new function. In Maple, to find the composition function is easy. For example, \( f(x) = 3x + 5 \) and \( g(x) = 3x^2 + 3x + 2 \), to find the composition \( f \circ g \), \( f \circ f \), \( g \circ f \), \( g \circ g \) and \( f^5(x) = (f \circ f \circ f \circ f \circ f)(x) \):

```maple
f:=x->3*x+5; g:=x->3*x^2+3*x+2;
plot(f); plot(g);
plot(f(g(x))); plot(g(f(x))); plot(g(g(x))); plot(f(f(f(f(x)))));
```

A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called odd if \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \). For example, \( f(x) = \ln(x + \sqrt{x^2 + 1}) \) is odd. Plot out the graph, we see that the graph of an odd function is symmetric about the origin \((0,0)\).

```maple
f:=x->ln(x+sqrt(1+x^2));
plot(f);
```

3. Piecewise-Defined Functions

We now define the function: \( g(x) = x - 1 \) for \( x \leq 2 \); \( g(x) = 5 - 4(x - 3)^2 \) for \( 2 < x < 4 \); otherwise \( g(x) = x + 1 \). This function is hard to understand without a picture, so we will have Maple plot it just as soon as we define it. The key point is that the function is given by different expressions for different parts of the domain. It is therefore piecewise-defined.
> g := x -> piecewise(x <= 2, x-1, x < 4, 5-4*(x-3)^2, x+1);
> plot(g,-2..8);

Notice the command syntax of defining a piecewise function: 
\[
\text{piecewise( } \text{cond}_1, f_1, \text{cond}_2, f_2, ..., \text{cond}_n, f_n, f_{\text{otherwise}} )\] 
where \( f_i \) – an expression, \( \text{cond}_i \) – a relation or a boolean combination of inequalities, \( f_{\text{otherwise}} \) – (optional) default expression.

4. Exercises

- 1. Plot a graph to find all \( x \) such that the inequality \( 5x^2 \leq 8x - 1 \) is true. Sketch the graph and indicate on the graph the region where the inequality is true. What would be the difference in the solution if the less than or equal to sign in the inequality were replaced with the less sign \(<\) ?

- 2. Plot the piecewise-defined function: 1). \( f(x) = -x^2 \) for \( x < -1 \); \( f(x) = x \) for \(-1 \leq x \leq 1 \); otherwise \( f(x) = -x^2 \). 2). \( f(x) = -1 \) for \( x \leq -1 \); \( f(x) = 3x + 2 \) for \( |x| < 1 \); otherwise \( f(x) = 7 - 2x \). 3). \( f(x) = x^2 - x^3 \) for \( x \leq -1 \); \( f(x) = 3x + 5 \) for \(-1 < x \leq 1 \); otherwise \( f(x) = x^4 + x^2 + x + 5 \). 4). \( f(x) = \sqrt{-x} \) for \( x < 0 \); \( f(x) = x \) for \( 0 \leq x \leq 2 \); otherwise \( f(x) = \sqrt{x - 2} \).

- 3. Given two functions \( f(x) = x^4 + 1 \) and \( g(x) = x^2 + 1 \), find the functions \( f \circ g \), \( f \circ f \), \( g \circ f \), \( g \circ g \) and \( f \circ (f \circ f \circ f \circ f)(x) \).

- 4. Define functions \( f(x) = \sqrt{\frac{x^2 + 5}{x^2 - 1}} \) and \( g(x) = \sqrt{\frac{x + 1}{x - 2}} \), find a decimal value of
  a) \( (f + g)(5) \); b) \( (f / g)(3) \); c) \( (fg)(4) \).
  (Hint: a) \( f(5) + g(5) \); b) \( \frac{f(3)}{g(3)} \); c) \( f(4) g(4) \))

- 5. Define \( f(x) = x^2 \) and \( g(x) = \sin(x) \)
  a) Define \( s(x) \) as the sum of \( f(x) \) and \( g(x) \), plot \( f(x) \), \( g(x) \) and \( s(x) \) in the same window.
  b) Define \( p(x) \) as the product of \( f(x) \) and \( g(x) \), plot \( f(x) \), \( g(x) \) and \( p(x) \) in the same window.
  c) Define \( c(x) \) as the composition \( f(g(x)) \), plot \( f(x), g(x) \) and \( c(x) \) in the same window.
  d) From the plots, can you see how \( f(x) \), \( g(x) \) are used to form each of \( s(x) \), \( p(x) \) and \( c(x) \)?
• 6. Find the approximate range of \( f(x) = \frac{40}{11x^2 - 42x + 48} \).

• 7. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be given by \( g(x) = \frac{x}{x^2 + 1} \) and \( f(x) \) defined as \( f(x) = x - 1 \) for \( x < 1 \), otherwise \( f(x) = x^2 + x \).
   Find the function (expressed by an analytic formula) \( g \circ f \).

• 8. A function \( f : \mathbb{R} \to \mathbb{R} \) is called even if \( f(-x) = f(x) \) for all \( x \in \mathbb{R} \). Plot the graph of the following function to see if it is an even function. The graph of an even function is symmetric along the \( y \)-axes.
   \[ f(x) = \ln^2 \left( \frac{x-1}{x+1} \right) \text{ for } 1 < |x|, \text{ otherwise } f(x) = 0. \]

• 9. Use the command \( \text{plot}(f1,x) \) to plot the following functions (view the details)
   a) \( f(x) = (256 - x^4)^{\frac{1}{4}} \);
   b) \( f(x) = \sin \left( \frac{1}{x} \right) \) for \(-0.1 < x < 0.1\) (what is happening near \( x=0 \)? \( \text{plot}(f, -0.1..0.1) \) or \( \text{plot}(f(x),x=-0.1..0.1) \));
   c) Plot the piecewisely defined functions in the question 2. View the details by near the ’contact’ points.

• 10. Consider the following function \( f(x) = \frac{\sin(x)}{x} \) for \( x < 0 \); \( f(x) = 1 \) for \( x = 0 \);
   and \( f(x) = \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}}}} \) for \( 0 < x \leq a \), where \( a \) is the root of the equation \( \frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}} = 0 \)
   a) Compute the point \( a \);
   b) Plot the function \( f(x) \).

**Lab 10. Inverse Functions**

In this Lab we will study inverse functions from the point of view of the cancelation equations, with the help of graphs.

1. Theory of Inverse Functions
   Given a function, how do you find the inverse function? First, you should make sure the function you are dealing with is one-to-one; that is, every \( y \)-value correspond to at most one \( x \)-value.
Let’s make three plots of \( \cos(x) \), one plot for each of the following domains: \( x = -10..10 \), \( x = -\frac{\pi}{2}..\frac{\pi}{2} \), and \( x = 0..\pi \). Use these plots to answer the following questions: Does \( \cos(x) \) have an inverse function, if \( x \) is allowed to be any number? Does it have it if \( x \) is restricted to the interval \( (-\frac{\pi}{2}, \frac{\pi}{2}) \)? What if \( x \) is allowed to be any number between 0 and \( \pi \)?

\[
> \text{plot}(\cos(x), x=-10..10);
> \text{plot}(\cos(x), x=-\pi/2..\pi/2);
> \text{plot}(\cos(x), x=0..\pi);
\]

From the graphs, we see that only in the last case, i.e. \( x = 0..\pi \), the function \( \cos(x) \) is monotonic, thus it has an inverse function.

Now, let us assume we have a one-to-one function. How do we find its inverse? If the graph of a function \( f \) is defined as a collection of ordered pairs \((x, f(x))\), then the graph of \( g \), the inverse of \( f \), is defined as the collection of ordered pairs \((f(x), x)\). Thus, reversing the position of the ordinate and abscissa for the function generates the graph of the inverse. Hence, the graphs of \( f \) and its inverse are reflections of each other across the line \( y = x \).

Operationally, two functions \( f \) and \( g \) are declared to be related as inverse one to another if for all \( x \) in the domain:

\[
\begin{align*}
  f(g(x)) &= x \\
  g(f(x)) &= x.
\end{align*}
\]

Algorithmically, the rule (formula) for \( g(x) \), the inverse of \( f \), is computed by solving the equation \( f(x) = y \) for \( x \), and then switching the letters \( x \) and \( y \). How does one do this in Maple? We will illustrate by examples.

2. Examples

Function \( f(x) = \frac{4x}{3+x} \) is not defined at \( x = -3 \), but it is one-to-one.

\[
> f := x -> 4*x/(3+x);
> \text{plot}(f, -\infty..\infty);
\]

This will produce a less-than-perfect graph, but perhaps it’s good enough to confirm that \( f \) is one-to-one. By the way, if you see a vertical line running through \( x = -3 \), it’s not part of the graph. It’s an error produced by Maple’s ‘join up the dots’ plotting routine.

We solve the equation \( f(x) = y \) for \( x \), Maple returns the answer \( \frac{-3y}{y-4} \).

\[
> \text{solve}(y=f(x), x);
\]

Now we define the inverse function for \( f \). It is simply this answer, but with \( x \) replacing \( y \):

\[
> g := x -> -3*x/(x-4);
\]

Finally, check that \( f \) and \( g \) are inverses by applying the cancelation equations:

\[
> f(g(x));
> \text{simplify}(\%);
\]
Next example: On one set of axes, plot \( f(x) = 1 + x^2 \), \( 0 \leq x \), the line \( y = x \), and the graph of the function whose ordered pairs are the inverse of those for \( f(x) \).

To plot the graph of these reversed ordered pairs, use a parametric representation.

\[
\begin{align*}
> & f := x \mapsto 1 + x^2; \\
> & \text{plot( }\{[x, f(x), x=0..3], [x, x, x=0..8], [f(x), x, x=0..3]\}, \text{scaling=constrained});
\end{align*}
\]

The graphs seem to be mirror images across the line \( y = x \). To verify that we have graphed the inverse of \( f \), we need to apply the algorithm for computing an inverse.

\[
\begin{align*}
> & q := \text{solve}(f(x) = y, x); \\
> & \text{This resulted in two possible expressions for } x. \text{ Since the domain of } f(x) \text{ is the set } \{x \mid 0 \leq x \}, \text{ we need to pick the first branch in } q, \text{ that is, the one that is always positive. Switch the letters in this expression to form the rule for } g(x), \text{ the inverse of } f(x). \\
> & g := \text{subs}(y=x, q[1]); \\
> & \text{Finally we can verify that the above } f \text{ and } g \text{ are inverses by showing that they satisfy the cancelation equations:} \\
> & q1 := \text{subs}(x=g, f(x)); \\
> & q2 := \text{subs}(x=f(x), g);
\end{align*}
\]

Here notice that \( f \) is defined as a function and \( g \) is defined as an expression. Also notice that \( \sqrt{x^2} \) is not simplified as \( x \). Maple will not simplify \( q2 \) to \( x \) because such a transformation is only correct if \( 0 \leq x \). While we know that this is so, Maple does not, and hence will not carry out the simplification.

\[
\begin{align*}
> & \text{assume}(x>0); \\
> & \text{simplify} \left( \text{subs}(x=f(x), g) \right);
\end{align*}
\]

The tilde (\(^\sim\)) attached to the \( x \) is a reminder that the result just produced is correct only in light of the assumption made about \( x \). To remove the assumption on \( x \), use the same technique as for removing values assigned to a variable.

\[
\begin{align*}
> & x := 'x';
\end{align*}
\]

Lastly, we will look at the inverse function for the \( \tan \) function on the domain \(-\pi < x < \pi\). It is usually called \( \arctan \). Try the following in Maple:

\[
\begin{align*}
> & \tan(\Pi/4); \\
> & \arctan(\%);
\end{align*}
\]

The first answer should be 1 and the second answer should be \( \frac{\Pi}{4} \). This is what you would expect, using the cancelation equations. This tells us that \( \pi = 4 \arctan(1) \).

\[
\begin{align*}
> & \Pi = 4 * \arctan(x);
\end{align*}
\]

But now here comes the really interesting part. Did you ever wonder how it happens that people know the value of the number \( \pi \) up to many digits of accuracy? You could try to measure both the circumference and the diameter of a circle and divide
the one by another. The answer should be $\pi$, but such measurements are bound to be inaccurate. In the question 2 of the following exercises, there is a way that involves no measurements and has nothing to do with circles.

3. Exercises

- 1. Following the steps in the examples, find the inverse of the function $f(x) = \sqrt{x^3 + 1}$ on the domain $-1 \leq x$. Check your answer using the cancelation equations.

- 2. Find the tenth-order Taylor polynomial, expanded about $x = 0$, for the function $f(x) = 4\arctan(x)$. Substitute $x = 1$ into this polynomial. What do you get? Did you get very close to the accepted value of $\pi$? Try increasing the order of the polynomial, to see if you can do better. By the way, the first eight digits of the accepted value of $\pi$ are 3.1415926, and you will need an enormous Taylor polynomial to get even a few digits correct. See how close you can come before Maple runs out of memory or you run out of time. Report the order of the highest-order Taylor polynomial you were able to try, and the approximate value for $\pi$ that resulted. (Hint: Maple command $Tn:=\text{taylor}(expr, x=a, n+1);$ will give $Tn$ the $n^{th}$-order Taylor polynomial of $expr$ about $a$. Maple command $Pn:=\text{convert}(Tn, \text{polynom});$ converts $Tn$ to a Maple polynomial (drops the $O(x^{n+1}))$).

- 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -x^2$ for $x < 0$; $f(x) = x$ for $x$ in $[0,1)$; otherwise $f(x) = 2x - 1$.
   a) verify if the function $f$ is one-to-one;
   b) find the range of $f$;
   c) find a formula for the inverse of $f$, i.e. $f^{(-1)}$.

- 4. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x + 1$ for $x < 1$; $g(x) = 3$ for $x = 1$; otherwise $g(x) = x^2 + 2x + 1$.
   a) verify if the function $f$ is one-to-one;
   b) find the range of $f$;
   c) find a formula for $f^{(-1)}$ (Hint: use the graph of $f$).

- 5. Check if the function $f(x) = \frac{x}{1+x^2}$ is one-to-one.
• 6. Show that the functions \( f(x) = e^x \) and \( g(x) = \ln(x) \) are inverse functions by showing that the compositions \( e^{\ln(x)} \) and \( \ln(e^x) \) both simplify to \( x \).

### Lab 11. Limits of Functions

We will develop several different methods in Maple to evaluate the limits of functions.

1. Guess the limit from its neighborhood behavior

Try the following:

```maple
> f:=x->sin(x)/x;
> f(0);
```

Why does Maple complain? This function is undefined at \( x = 0 \). We want to find the limit when \( x \to 0 \), which means we want to find the value of this function as \( x \) approaches 0. Let’s try the approaching procedure with the help of Maple.

```maple
> f(0.1); f(0.001); f(0.0001); f(0.00001);
```

Now, can you guess the limit as \( x \) approaches 0?

Next, let’s take a look at the following two functions \( f_1 \) and \( f_2 \).

```maple
> f1:=x->1/2*(-1+sqrt(1+4*x^2))/x;
> f2:=x->1/2*(-1-sqrt(1+4*x^2))/x;
> plot(f1,-3..3);
> plot(f2,-3..3,-10..10);
```

Each formula \( f_1 \) and \( f_2 \) contains an \( x \) in the denominator, suggesting that perhaps \( x = 0 \) should be excluded from the domains. But the graph of \( f_1 \) seems to be untroubled by this glitch, whereas the graph of \( f_2 \) has a vertical asymptote at \( x = 0 \). The reason for these observations is hidden in the limiting behaviors of \( f_1 \) and \( f_2 \), and that is why the concept of limit is so important in the calculus. When plotting, Maple computes values of the function to be graphed, and then ‘connects the dots’. So, what numbers could Maple have computed prior to plotting \( f_1 \)? What would have happened if Maple had used \( x = 0 \) itself as one of the values?

```maple
> f1(0);
```

Without doubt, formula \( f_1 \) is not defined at \( x = 0 \). So, Maple’s \texttt{plot} command did not compute \( f_1 \) exactly at \( x = 0 \), but used values of \( x \) near \( x = 0 \). What values will formula \( f_1 \) produce for \( x \)’s near \( x = 0 \)? We’ll use a for-loop for the repetitious evaluation of the result of substitutions into \( f_1 \) of ever smaller values of \( x \). The loop ends with \texttt{od}, which is clearly \texttt{do} spelled backwards.

```maple
> for k from 1 to 5 do
> 0.1^k, evalf(f1(0.1^k));
> od;
```
As the value of $x$ became smaller, the value of $f_1$ also became smaller. In fact, by the time $x$ was $1/100000$ Maple computed $f_1$ to be zero in 10-digit arithmetic. But is that reasonable? Can Maple compute more accurately?

```maple
> for k from 1 to 5 do
  > 0.1^k, evalf(f1(0.1^k),20);
> od;
```

The `evalf` command takes a second argument, an integer, that specifies the number of digits the floating-point evaluation is to contain. At this extended precision we see that when $x = 1/100000$, a more reasonable value for $f_1$ is $1/100000$. There are two immediate conclusions to be drawn. First, it appears that for $x$ small enough, $f_1$ behaves like the function $y = x$. That observation seems consistent with the graph of $f_1$, but the verification of this ‘local linearity’ is at the very heart of the calculus and will be explored in subsequent labs. Second, since $f_1$ gets small as $x$ gets small, Maple was not remiss when it connected the dots in the graph of $f_1$. The single point $(0, 0)$ should be omitted from the graph. Even if Maple had done this, it would have been impossible to detect the missing point on the graph. Hence, we would likely say that the limiting value of $f_1$ as $x$ gets near zero is itself zero.

2. Graphic Technique for Computation of Limits

From the graphs in the previous section, you probably already get a hint: we can use Maple to plot the function and read the limit off the plot. First, let’s try this with $f(x) = \frac{\sin(x)}{x}$.

```maple
> f := x->(x^2-4)/(x-2);
> f(2);
> plot(f,1..3);
> plot(f,1.999..2.001);
```

The function is undefined at $x = 2$. If you read right, $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$. Indeed, the fraction in $f$ simplifies to

```maple
> simplify(f(x));
```

at all points where $x$ is not 2. It is from this reduced form of the fraction that we determine the limit to be 4 when $x$ gets near 2.

Some limits don’t exist, but probably the left-hand and right-hand limits exist. We can also read these limits off the graphs. For example, the function $\text{abs}(x)$ is used
by Maple to compute the absolute value of $x$, denoted in standard mathematical notation by $|x|$.

```maple
> plot(abs(x),x=-2..2);
```

Now divide this function by $x$ to get a new function $\frac{|x|}{x}$. Find the left-hand and right-hand limits of the function $\frac{|x|}{x}$ as $x \to 0$.

```maple
> plot(abs(x)/x,x=-0.1..0.1);
```

From this graph we can guess that the left-hand limit is $-1$ and the right-hand limit is $1$. Next, try to find the left-hand and right-hand limits from the graph for $f_2$ as $x \to 0$.

```maple
> plot(f2,-3..3,-infinity..infinity);
```

Thus, the left-hand limit is $\infty$ and the right-hand limit is $-\infty$. You can try this method to find the limits of $f_2$ as $x \to \infty$ and as $x \to -\infty$. But it is not obvious. It may take your time before you figure it out.

3. Maple Command for Limits

In fact, the idea of limit is a built-in function in Maple. Furthermore, Maple can compute left-hand and right-hand limits as well. Let’s verify the limits we guessed in the previous sections.

```maple
> limit(sin(x)/x,x=0);
> limit(f1(x),x=0);
> limit((x^2-4)/(x-2),x=2);
> limit(abs(x)/x,x=0);
> limit(abs(x)/x,x=0,left);
> limit(abs(x)/x,x=0,right);
> limit(f2(x),x=0);
> limit(f2(x),x=0,left);
> limit(f2(x),x=0,right);
```

With this limit command in Maple, many problems become easier.

```maple
> limit(f2(x),x=-infinity);
> limit(f2(x),x=infinity);
> limit(f1(x),x=infinity);
> limit(f1(x),x=-infinity);
```

Let’s try more questions.

```maple
> limit((1-cos(x))/x,x=0);
> limit((1+1/x)^x,x=infinity);
> limit((1+1/x)^x,x=0);
```

However, Maple may not be able to solve every question. In such a case you will need to use your own skills to simplify the expression before you ask Maple to find the limit.

```maple
> e1:=((-2)^x+3^x)/((-2)^(x+1)+3^(x+1));
```
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> limit(e1,x=infinity);
> e2 := ((-2/3)^x+1)/(-2*(-2/3)^x+3);
> limit(e2,x=infinity);

4. Exercises

- 1. Use Maple to evaluate the limit of \( \frac{1-\cos(x)}{x} \) as \( x \to 0 \) by three different methods (notice this function is undefined at \( x = 0 \)). First, evaluate this function at \( x = 0.1 \), \( x = 0.01 \), \( x = 0.001 \) and \( x = 0.0001 \), use these values to guess the limit. Next, use Maple to plot the function and read the limit off the plot. Finally, use Maple’s limit command to evaluate the limit exactly.

- 2. Plot the functions \( 1 - x^2 \) and \( |1 - x^2| \) on a single set of axes. Now define the new function \( f(x) = \frac{|1-x^2|}{1-x^2} \) and compute its limits as \( x \to 1 \) from the left and as \( x \to 1 \) from the right. Provide a plot of \( f(x) \) illustrating the behavior as \( x \to 1 \) (it may improve the plot if you specify `discont=true` as an option in the Maple `plot()` command). What happens when you try to compute the limit at \( x = 1 \) without specifying a direction (left or right)? Why?

- 3. Investigate using graphical methods the limit of each of the following functions \( f(x) \) as \( x \to 1 \). Does the limit exist, and if so, what is it? Compare your answer to that obtained using limit command. Can you ensure that \( f(x) \) is within 0.1 of some number \( L \) by taking \( x \) close enough to 1? What about 0.01? What about an arbitrary \( \varepsilon > 0 \)?
  a) \( f(x) = x^2 - 3x; \)
  b) \( f(x) = \frac{\sin(x-1)}{1-x^2}; \)
  c) \( f(x) = \text{Heaviside}(\sin(\frac{1}{x})); \)

- 4. Investigate \( \lim_{x \to 0^+} x^{\sin(x)} \) graphically. How small must \( x \) be (with \( x > 0 \)) to ensure that \( x^{\sin(x)} \) is between 0.9 and 1.1? What about between 0.99 and 1.01?

- 5. Investigate the following limits graphically using “chart” (as we did in section 1). Determine the limit to four decimal places. Compare to Maple’s answer for the limit.
  a) \( \lim_{x \to 1} \frac{-1+x^2}{x^2+\sqrt{x}-2}; \)
  b) \( \lim_{x \to 0} \cos(2x)^{\frac{1}{x^2}}. \)
6. A student tried to investigate $\lim_{x \to 0} \sin(\frac{\pi}{x})$ using “chart” with $x = 0.1, -0.1, 0.01, -0.01, 0.001, -0.001,$ etc. What went wrong with this approach?

7. This exercise shows some of the limitations of graphical techniques for limits. Consider $\lim_{x \to 0^+} f(x),$ where $f(x) = \frac{x}{\sin(\frac{1}{x})}.$
   a) Plot $f(x)$ on the intervals $[0,0.5],$ $[0,0.05]$ and $[0,0.005].$ What does the limit appear to be in each case? Compare to Maple’s answer for the limit.
   b) Can you find an $x$ for which $|f(x)| \leq 0.01?$ Hint: you’ll need to increase “Digits” to at least 50 for this one. Can you explain what is happening in (a)?
   c) Now consider $g(x) = |\sin(\frac{1}{x})|^{\frac{1}{x^2}}.$ Plot $g(x)$ on the same intervals used in (a). What does $\lim_{x \to 0^+} g(x)$ appear to be?
   d) What is $g(x)$ when $\frac{1}{2}$ is an odd multiple of $\frac{\pi}{2}?$ What does this say about the limit in (c)?

8. Investigate the following limits graphically, using transformations where appropriate.
   a) $\lim_{x \to \infty} x(\frac{1}{2});$
   b) $\lim_{x \to 1^+} \frac{\sqrt{x} - 1}{x - 1};$
   c) $\lim_{x \to -\infty} (x^2 + 1)(\frac{1}{2}) + x^3.$

9. a) Which of the following expressions is difficult to calculate accurately when $x$ is close to 0 because of roundoff error? i) $\sqrt{1+x^2} - \sqrt{1-x^2}$; ii) $\frac{\sin(x)}{x+3x^2}$.
   b) How large must “Digits” be in order to have Maple’s value for this expression with $x = 0.00001$ be within 0.001 of its limit as $x \to 0?$ Within 0.00001? Do you think it is ever within $10^{-20}?$

10. Evaluate the following limits:
   a) $\lim_{x \to -4} \sqrt{x} + \sqrt{-x};$ b) $\lim_{x \to 0^-} \sqrt{-x};$ c) $\lim_{t \to 1} \frac{t+1}{t^3-1};$ d) $\lim_{t \to 4} \frac{t-4}{t^2-3t-4};$
    e) $\lim_{h \to 0} \frac{(1+h)^2 - 1}{h};$ f) $\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 + 3x + 2};$ g) $\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 + 3x + 2};$ h) $\lim_{t \to 6} \frac{17}{(t-6)^2};$
    i) $\lim_{x \to -6^+} \frac{x}{x+6};$ j) $\lim_{s \to 16} \frac{4 - \sqrt{s}}{4};$ k) $\lim_{v \to 2} \frac{v^2 + 2v - 8}{v^2 - 16};$ l) $\lim_{x \to 8} \frac{4x - x^2}{x^2 - 16};$
    m) $\lim_{x \to -9^+} \sqrt{x} + [x + 1],$ see below for the definition of the function $[x];$ n)
\[ \lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x - \frac{1}{x}}; \]

\[ \lim_{x \to -2} \frac{\sqrt{x+2} - \sqrt{2x}}{x - 2}; \]

The function \( f(x) = [x] \), called greatest integer function, is defined by \( f(x) = n \) such that \( n \leq x \). The greatest integer function can be defined in Maple by using the following command line:

\[ \text{gint1:=x->piecewise(x>=0,round(x-0.5),x<0,-round(-x+0.5))}; \]

or

\[ \text{gint2:=x->piecewise(x>=0,trunc(x),x<0,} \]
\[ \quad \text{trunc(x-trunc(x)+1)+trunc(x)-1);} \]

Check if these two functions satisfy the definition of \([x]\) and how they work in various situations.

- 11. Compute the limits:
  a) \( \lim_{x \to \infty} \frac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}} \), apply \textit{ifactor} (try \textit{?ifactor} to learn the command \textit{ifactor}()) to the numerator and denominator;
  b) \( \lim_{x \to 2} \frac{x^3-2x^2-4x+8}{x^4-8x^2+16} ; \)
  c) \( \lim_{x \to 2} \frac{(x^2-x-2)^{20}}{(x^4-12x+16)^{10}} ; \)
  d) \( \lim_{x \to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} ; \)
  e) \( \lim_{x \to \pi/6} \frac{\sin(x)^2+\sin(x)-1}{2 \sin(x)^2-3\sin(x)+1} ; \)
  f) \( \lim_{x \to \pi/4} \frac{1-\cot(x)^3}{2-\cot(x)-\cot(x)^3} ; \)
  g) \( \lim_{x \to \infty} \left( \frac{3x^2-x+1}{2x^2+x+2} \right)^{\frac{x^3}{x+1}} ; \)
  h) \( \lim_{x \to \pi/4} \sin(x)^{\tan(x)} . \)

**Lab 12. Continuity of Functions**

We will explore with the help of Maple several different kinds of discontinuity. We will learn Maple commands \textit{iscont}(), \textit{discont}() and the plot option \textit{discont=true}.

1. Definition of Continuity

First, let’s review the idea and the definition of function continuity. The idea of continuity comes from geometry. Thus we could use Maple to plot out the graph of the function to see if the function is continuous or not. Let \( f : \text{Dom}(f) \to \mathbb{R} \) be a
function and \( x_0 \) is a point in the \( \text{Dom}(f) \). There are two possibilities: either \( x_0 \) is an accumulation point of \( \text{Dom}(f) \) or not. If \( x_0 \) is an isolated point (not an accumulation point) in \( \text{Dom}(f) \), we assume the function \( f \) is always continuous at \( x_0 \). However, if \( x_0 \) in \( \text{Dom}(f) \) is an accumulation point of \( \text{Dom}(f) \) then we need the precise definition: \( f(x) \) is continuous at \( x_0 \) if and only if \( \lim_{x \to x_0} f(x) = f(x_0) \).

If the function \( f(x) \) is continuous at \( x_0 \) then we say that \( x_0 \) is a continuity point of \( f(x) \), we say that \( f(x) \) is continuous if it is continuous at every point \( x \) in \( \text{Dom}(f) \).

With the help of the definition and the computation of limit, it’s easy for us to understand the continuity.

\[
> f := x \rightarrow x^3 + 2x - 5;
> \text{plot}(f, -\infty..\infty);
\]

From the graph of function \( x^3 + 2x - 5 \), it seems to be continuous everywhere.

\[
> \text{readlib(iscont)}: \text{iscont}(f(x), x = -\infty..\infty);
\]

With the help of Maple command \text{iscont()}\), we can confirm that the above function is continuous. For the details of the command \text{iscont()}, please try \? \text{iscont}\. Execute the following Maple command one by one. After these, you will have some ideas about one important elementary function (\text{tan} function) and the Maple command \text{iscont()}

\[
> \text{iscont}(\sin(x), x = -\infty..\infty);
> \text{iscont}(\tan(x), x = -\infty..\infty);
> \text{plot}(\tan(x), x = -\infty..\infty);
> \text{plot}(\tan(x), x = -\Pi/2..\Pi/2, -10..10);
> \text{iscont}(\tan(x), x = -\Pi/2..\Pi/2);
> \text{iscont}(\tan(x), x = -\Pi/2..\Pi/2, 'closed');
\]

2. Discontinuity Points of a Function

Let us look carefully at the definition of the continuity. For a point \( x_0 \) to be a continuity point of \( f(x) \), first it is required that \( x_0 \) is in \( \text{Dom}(f) \), then it is required that the limit of \( f(x) \) as \( x \to x_0 \) exists, i.e. \( \lim_{x \to x_0} f(x) \) exists, and finally this limit should be equal to the value of the function at the point \( x = x_0 \), i.e. \( \lim_{x \to x_0} f(x) = f(x_0) \).

Sometimes \( \lim_{x \to x_0} f(x) \) exists, while the \( x_0 \) is not in the domain of the function. This is called the removable discontinuity. From the graph of this function, it seems to be continuous. Take a look at the following example.

\[
> f := x \rightarrow (x^2 - 4)/(x - 2);
> \text{limit}(f(x), x = 2);
> \text{plot}(f, -3..3, \text{discont} = \text{true});
> \text{readlib(discont)}: \text{discont}(f(x), x);
\]

For the other discontinuities, we can detect them from the graphs.

\[
> f := x \rightarrow \text{piecewise}(x \leq 3, x^2, x > 3, x^3);
\]
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> plot(f,2..4,discont=true);
For the details of the plot option discont=true, please try ? plot[options]. As $x \to 3$, the left-hand and right-hand limits of this function exist, while they are not the same. Hence, the limit of this function as $x \to 3$ doesn’t exist. In fact, $x = 3$ is a discontinuous point of the function. You can see it from the graph.

> limit(f(x),x=3,left);
> limit(f(x),x=3,right);
> limit(f(x),x=3);
> discont(f(x),x);

Some function has the limit, i.e. $\lim_{x \to x_0} f(x) = L$ but the limit $L$ is not the same as the value of the function at this point $f(x_0)$. This kind of discontinuity cannot be detected from the graph.

> f:=x->piecewise(abs(x)>0,abs(x),1);
> f(0);
> limit(f(x),x=0);
> discont(f(x),x);
> plot(f,discont=true);

Take a look at $\tan$ function, at $x_0 = (k + \frac{1}{2})\pi$ where k is an integer, $\tan(x_0)$ is undefined, while $\lim_{x \to x_0^-} \tan(x) = \infty$ and $\lim_{x \to x_0^+} \tan(x) = -\infty$. Hence $\tan$ function is discontinuous at $x_0 = (k + \frac{1}{2})\pi$, for any integer k. This kind of discontinuity can be seen from the graph.

> plot(tan(x),x=-10..10,y=-10..10, discont=true);

3. Exercises

- 1. Define the function piecewisely:

  $f(x) = \sqrt{-x}$ for $x < 0$; $f(x) = 3 - x$ for $0 \leq x < 3$; otherwise $f(x) = (x - 3)^2$.

  a) Evaluate each limit, if it exists: $\lim_{x \to -0} f(x)$, $\lim_{x \to -0^-} f(x)$, $\lim_{x \to -0^+} f(x)$, $\lim_{x \to 3^-} f(x)$, $\lim_{x \to 3^+} f(x)$, and $\lim_{x \to 3} f(x)$;

  b) Plot the graph of $f(x)$;

  c) Find discontinuity points of $f(x)$ using the command discont( ).

- 2. Consider the following function:

  $f(x) = \frac{\sin(x)}{x}$ for $x < 0$; $f(x) = 1$ for $x = 0$; $f(x) = \sqrt{\frac{1}{x} + \frac{1}{x} + \frac{1}{x} - \sqrt{\frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}}}$ for $0 < x \leq a$. 
Compute the one-side limits at \( x = 0 \) in order to determine if the function \( f(x) \) is continuous.

- 3. Classify the discontinuity points of the following functions
  a) \( f(x) = \frac{x}{\sin(x)} \);
  b) \( f(x) = \frac{1}{\ln(x)} \);
  c) \( f(x) = e^{(x+\frac{1}{x})} \).

- 4. The greatest integer function \( f(x) = [x] \) is defined by \( [x] = \) the largest integer that is less than or equal to \( x \). Classify the discontinuity points and sketch the graphs of the following functions
  a) \( f(x) = x - [x] \);
  b) \( f(x) = x [x] \);
  c) \( f(x) = \frac{x}{[x]} \).

  Hint: find the Maple definition for the greatest integer function in the previous lab.

**Lab 13. Elementary Functions**

In this Lab, we will explore the seven types of elementary functions discussed in the textbook. Use the technique in Maple, we will discuss their domain, range, continuity and some basic properties. Lastly, we will create an animated graph for some functions.

1. Elementary Functions

   **Type I: Polynomials and Rational Functions**

   A polynomial \( P \) is a function of the type \( P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, a_n \) is not zero. The number \( n \) is called degree of the polynomial \( P \). It is clear that the domain of a polynomial is \( \mathbb{R} \) and it’s continuous everywhere. Take a look at the graph of this polynomial: \( P(x) = 4 x^4 - 3 x^2 + 5 x - 10 \).

   \[
   P := x \rightarrow 4 \times x^4 - 3 \times x^2 + 5 \times x - 10; \text{plot}(P, -\infty \ldots \infty); \]

   From this graph, we know that there are two zero points, i.e. the equation \( 4 x^4 - 3 x^2 + 5 x - 10 = 0 \) has two real roots. While the basic theorem of algebra says: the equation \( P(x) = 0 \) has \( n \) roots(including real and complex roots counted with multiplicity), where \( n \) is the degree of the polynomial \( P(x) \). Thus for the above polynomial, there
are two other complex roots. You can get the real roots by using command \texttt{fsolve()}
(for the details, please try \texttt{?fsolve}).
\begin{verbatim}
> plot(P,-8..8,-10..10);
> take a closer look at the two zero points
> fsolve(P(x)=0,x);
\end{verbatim}
A rational function \( R(x) \) is a quotient of two polynomials, i.e. \( R(x) = \frac{P(x)}{Q(x)} \), where
\( P \) and \( Q \) are two polynomials. The domain of \( R(x) \) is the set of all real numbers \( x \)
such that \( Q(x) \) is not zero. And it is continuous on its domain. Consider the function
\[ f(x) = \frac{x^3+4}{x^2+2x-3}. \]
\begin{verbatim}
> quo(x^3+4,x^2+2*x-3,x);
\end{verbatim}
The command \texttt{quo(poly1,poly2,var)} accepts two polynomials as arguments and
divides the second polynomial into the first. It reports the result but does not report
the remainder. For this example, Maple returns the answer: \( x - 2 \).
\begin{verbatim}
> rem(x^3+4,x^2+2*x-3,x);
\end{verbatim}
This gives the remainder term. For this example, Maple returns the answer: \( -2 + 7x \).
These Maple responses are meant to tell us that:
\[ \frac{x^3+4}{x^2+2x-3} = x - 2 + \frac{-2+7x}{x^2+2x-3}. \]
\begin{verbatim}
> f:=x->(x^3+4)/(x^2+2*x-3);
> plot(f,-infinity..infinity);
> readlib(discont): discont(f(x),x);
> solve(x^2+2*x-3=0,x);
> plot(f,-10..10,-20..10,discont=true);
\end{verbatim}
There are two points \( x_0 = -3 \) and \( x_0 = 1 \) making the denominator to be zero. Thus
the domain of this rational function \( f(x) \) is \(( -\infty, -3 ) \cup ( -3, 1 ) \cup ( 1, \infty )\), where “\( \cup \)”
means union. In fact, \( x = -3 \) and \( x = 1 \) are two vertical asymptotes. And from
the graph, we see no horizontal asymptote. We will learn this later on.

\textbf{Type II: Functions}
A power function is a function of the type \( f(x) = x^\alpha \), where \( \alpha \) is a real number and \( x \) is positive. Sometimes the domain of \( f \) can be extended to negative numbers, even
including \( x = 0 \). Power functions are continuous on their domains.
\begin{verbatim}
> plot({x,x^(1/2),x^(1/3),x^2,x^3},x=-3..3);
\end{verbatim}
Can you recognize which graph is for which function? Hint: first find out the domain
and range for each individual function, then find out the differences between \( y = x^2 \)
and \( y = x^3 \), \( y = x^{(\frac{1}{2})} \) and \( y = x^{(\frac{1}{3})} \).

\textbf{Type III: Exponential Functions}
An exponential function is a function of the type \( f(x) = a^x \), where \( a > 0 \) is a constant.
There is a special exponential function \( e^x \). In Maple, it is denoted by \texttt{exp(x)}.
\begin{verbatim}
> plot( {(1/2)^x,(1/4)^x,1^x,1.5^x,2^x, exp(x),4^x,10^x},x=-1..1,
y=0..4);
\end{verbatim}
There is an obvious observation: the functions values are all positive. Again, can you
point out which graph is for which function?
**Type IV: Logarithmic Functions**

A logarithmic function is a function of the type \( f(x) = \log_a(x) \), where \( a \) is positive and not equal to 1, \( x > 0 \). The function \( \ln(x) \) is simply called the natural logarithm.

![Plot of logarithmic functions](image1)

Which graph is for which function?

Exponentials and logarithms are inverses to each other for the same bases. They satisfy the cancelation equations. Their graphs are reflections of each other across the line \( y = x \). For example, \( e^x \) and \( \ln(x) \), \( 2^x \) and \( \log_2(x) \), \( \left(\frac{1}{2}\right)^x \) and \( \log_{\frac{1}{2}}(x) \), \( 10^x \) and \( \log_{10}(x) \) etc. are pairs of inverse functions.

![Plot of exponential and logarithmic functions](image2)

**Type V: Trigonometric Functions**

The functions \( y = \sin(x) \), \( y = \cos(x) \), \( y = \tan(x) = \frac{\sin(x)}{\cos(x)} \), \( y = \cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} \), \( y = \sec(x) = \frac{1}{\cos(x)} \), \( y = \csc(x) = \frac{1}{\sin(x)} \). Sine function and cosine function have all the real numbers for their domain, and their range is \([-1,1]\). The domain of tan function includes all real numbers except \((k + \frac{1}{2})\pi\), where \( k \) is an integer, and the range is \( \mathbb{R} \). The domain of cot function includes all real numbers except \( k\pi \), where \( k \) is an integer, and the range is \( \mathbb{R} \). Then you can deduce the domain and the range of the sec function and csc function from the domain, range and zeros of sine function and cosine function. \( \text{Dom}(\sec) = \text{Dom}(\tan) \), \( \text{Dom}(\csc) = \text{Dom}(\cot) \), \( \text{Range}(\sec) = \text{Range}(\csc) = (-\infty, -1] \cup [1, \infty) \). Finally, they are all continuous on their own domain. We can see all these from their graphs.

![Plot of trigonometric functions](image3)

Now, we will define and plot two new expressions which are the squares of the sine and cosine functions:

\[
\text{sinSquared} := (\sin(x))^2; \quad \text{cosSquared} := (\cos(x))^2;
\]

Now, let’s try something interesting:

\[
\text{plot}(\text{sinSquared} + \text{cosSquared}, x=-4\pi..4\pi);
\]

**Type VI: Hyperbolic Functions**

...
The hyperbolic functions are defined by $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, and $\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.

Try to plot all the graphs by yourself, learn the domain, range and some basic properties of these function from the graphs. Also you could use the plotting in Maple to verify some important identities for hyperbolic functions.

**Type VII: Inverse Trigonometric Functions**

By reducing appropriately the domains of trigonometric functions, the restricted to these domains functions become one-to-one, so we can have their inverses. The inverse trigonometric functions are $y = \arcsin(x)$, $y = \arccos(x)$, $y = \arctan(x)$, $y = \arccot(x)$, $y = \arccsc(x)$, and $y = \text{arcsec}(x)$ etc. Plot the graphs to verify the reflection property across the line $y = x$, and the cancelation equations. Certainly, first of all, keep in mind their domains and ranges.

2. Animated Graphs

Consider the function $f(x) = A \sin(x + c)$, where $A$ and $c$ are constants. We wish to study what happens as we choose different values for these constants. To do this, try the following animation – be sure to load the full plots package first:

```maple
> with(plots):
> animate(A*sin(x),x=-4*Pi..4*Pi,A=-5..5, frames=64);
```

After executing the above command, a graph will appear. Click on the graph, the tool-bar for graphics will appear on the top of this window. Then click the play button in the tool-bar. You will see a movie whose frames are graphs of $f(x) = A \sin(x)$. In the first frame, $A$ has value $A = -5$, which increase during the movie until $A = 5$ in the final frame.

Fix $A = 1$ and make a movie in which $c$ varies from 0 to $2\pi$. As $c$ increases, in which direction does the graph move, left or right?

```maple
> animate(sin(x+c),x=-4*Pi..4*Pi,c=0..2, frames=64);
```

Suppose that a certain function has a form given by $f(x) = A \sin(x + c)$, the graph of $f$ crosses the $x$-axis at $x = \frac{\pi}{4}$, and $f(x)$ is positive when $x = 0$. Moreover, the $y$-value of the highest point on the graph is $y = 3$. Can you write down the exact formula for this function (in other words, find $A$ and $c$)? Then use Maple to plot it
in order to verify your answer. Try to answer this question before you move on to
the next step of this Lab.
> plot(3*sin(-x+Pi/4),x=-Pi..Pi);

3. Exercises

• 1. Plot the graph for \( f(x) = \frac{x}{1-x^2} \), find the domain and range of this function. Do you get any asymptote?

• 2. Use \texttt{quo} and \texttt{rem} to decompose the rational function \( f(x) = \frac{3x^2-1}{x+4} \).

• 3. Try to solve an equation involving trigonometric functions: \( \cos(2x) = \cos(x) \). Maple command \texttt{solve()} and \texttt{fsolve()} don’t help very much. Plot both \( \cos(2x) \) and \( \cos(x) \) on a single set of axes. Use the plot to write down all solutions to the above equation.

• 4. Follow the procedure described in section 1 for Type V functions to plot on separate graphs \( \tan(x)^2 \) (not \( \tan(x^2) \)), \( \sec(x)^2 \), and \( \sec(x)^2 - \tan(x)^2 \). What trig identity is at work in this case?

• 5. Similarly to question 4 try to verify all the trigonometric identities using plotting technique in Maple.

• 6. Plot each of the following functions or combinations of functions on the domain \([-5,5]\). Then answer the questions below.
  a) \( \cosh(x) \);  b) \( \cosh(-x) \);  c) \( \sinh(x) \);  d) \( \sinh(-x) \);  e) \( \cosh(x) + \sinh(x) \);  f) \( \cosh(x) - \sinh(x) \);
  g) \( \cosh(x)^2 - \sinh(x)^2 \);

  i) Compare plots (a) and (b). Is there any difference? Explain.
  ii) Compare plots (c) and (d). Is there any difference? Explain.
  iii) Do you recognize plots (e) and (f) as being the graphs of familiar functions? Which functions?
  iv) From (g), what does the combination \( \cosh(x)^2 - \sinh(x)^2 \) appear to equal? Using the above definitions for \( \sinh \) and \( \cosh \), work out what this combination should equal.
7. Equations involving hyperbolic functions cannot always be solved analyti-
cally. In such cases, we resort to graphical techniques. Use Maple’s plotting
abilities to find solutions to the following equations. When there are no solu-
tions, can you explain why not?
a) $1 + 2 \sinh(x) = \cosh(x)$;  
b) $\sinh(x) = \cosh(x)$;  
c) $x = \sinh(x)$;  
d) $x = \tanh(x)$.

8. Use Maple movies, describe how the shapes of the graphs of the following
functions change when the values of constants are changed. Describe also those
aspects of the shapes that do not change.
a) $f(x) = ax^2 + bx + c$ when $c$ changes. [Choose some particular values for
   $a$ and $b$ and let $c$ vary from frame to frame in the animation. Repeat once or
twice with different choices of $a$ and $b$ to make sure your choices don’t strongly
   affect your observations.]
b) $f(x) = ax^2 + bx + c$ when $a$ changes. [You’ll have to choose some specific
   values for $b$ and $c$ this time.] What happens when $a$ changes sign?
c) $f(x) = \cos(kx) + B$ when $B$ changes. Using this result and your answer to
   part (a), can you make a general remark about what happens to the graph of
   any function of the form $f(x) = g(x) + B$ when $B$ changes?
d) $f(x) = \cos(kx) + B$ when $k$ changes.

Lab 14. Slopes and Derivatives

In this Lab we will use Maple as an aid to understanding the concept of derivative
by looking at the slopes of secant lines and tangent lines.

1. Secant Lines and Tangent Lines

Consider a secant line joining two points $P = (x_0, f(x_0))$ and $Q = (x_0 + h, f(x_0 + h))$
on the graph of a function $f(x)$. The slope of the secant line is $q = \frac{f(x_0 + h) - f(x_0)}{h}$
(this expression is called a Newton quotient or difference quotient for $f$ at $x_0$,
and so the secant line itself has equation $y = f(x_0) + q(x - x_0)$. The tangent line
(if it exists and is non-vertical) has slope $m = \lim_{h \to 0} q = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$
so its equation is $y = f(x_0) + m(x - x_0)$. Note: be sure to make the following definitions
before assigning values to $f$, $x_0$, or $h$. If any of these names has already been assigned
a value in this session, you can unassign it:

> f := 'f';
If there are a lot of variables to unassign and none that you want to keep, an alternative is the `restart` command, which will undo everything and return Maple to its initial state.

```plaintext
> q := (f(x0+h)-f(x0))/h;
> secline := x -> f(x0)+q*(x-x0);
> m := limit(q, h=0);
> tanline := x -> f(x0)+m*(x-x0);
```

The answer for `m` is `m := D(f)(x0)`. As we will see, this is how Maple writes the derivative of the function `f` at `x0`. So, Maple knows the definition of derivative.

We can now choose a typical function `f` and some values for `x0` and `h`, and plot the function, the secant line and the tangent line together:

```plaintext
> f := x -> 2*x-x^3;
> x0 := 1;
> h := 1/2;
> plot({f(x), tanline(x), secline(x)}, x=0..2);
```

To see what happens as `h` approaches 0, we might try `h` = \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \) on the same plot. Here is a bit of a trick, to get these values of `h` as `kk` goes from 1 to 4:

```plaintext
> h := 1/2^kk;
> plot({f(x), tanline(x), seq(secline(x), kk=1..4)}, x=0..2);
```

We should also try negative `h` values (remembering to first unassign `kk` after the `seq`).

```plaintext
> kk := 'kk';
> h := -1/2^kk;
> plot({f(x), tanline(x), seq(secline(x), kk=1..4)}, x=0..2);
```

We could also try an animation:

```plaintext
> h := 'h';
> with(plots):
> animate({f(x), tanline(x), seq(secline(x), kk=1.4)}, x=0..2, h=-1/2..1/2);
```

2. Tangent Lines and the Derivative

We now want to think of the slope `m` as depending on `x0`, so let’s unassign the variable `x0`. In the next animation, Maple shows the function `f(x)`, its derivative (written as `D(f)(x)`), and its tangent lines, each frame using a different `x0`. Observe that the slope of each tangent line is the value of `D(f)` at the point of tangency. In particular, this is zero at the points where the tangent line is horizontal. To get both of those points in the picture, we will use this time the interval `-1.2..1.2`.

```plaintext
> x0 := 'x0';
> animate({f(x), D(f)(x), tanline(x)}, x=-1.2..1.2, x0=-1.2..1.2);
```

Now let’s plot this in one of the cases where the tangent line is horizontal. We will use `solve` (although this example is simple enough to do by hand). More difficult examples, where `solve` can’t find a solution, require `fsolve`. Note that, when used on
a polynomial, \texttt{solve} returns all of the roots. We can assign this collection of roots to a variable \( r \), and then refer to the individual roots as \( r[1], r[2], \) etc.

\begin{verbatim}
> D(f)(x);
> r:=solve(%,x);
> x0:=r[1];
> plot({f(x),D(f)(x),tanline(x)},x=-1.2 ..1.2);
\end{verbatim}

3. An Extra Touch

There’s still one improvement possible: a vertical line at \( x = x_0 \) to indicate the point of tangency (the tangent line is so close to the curve that it’s hard to tell exactly where they touch). First we will save the last plot by assigning it to a name. To avoid having to see the whole plot structure printed out, we’ll use : instead of ;.

\begin{verbatim}
> p1:=%:
\end{verbatim}

Next, we assign a plot of the vertical line to another name. The command \texttt{plot([[a,b],[c,d]])} draw a straight line joining points \( (a,b) \) and \( (c,d) \).

\begin{verbatim}
> p2:=plot([[x0,0],[x0,f(x0)]]):
\end{verbatim}

Finally, the display command will put both together.

\begin{verbatim}
> display({p1,p2});
\end{verbatim}

To make the last graph more understandable, we will use \texttt{textplot} command to add some text labels to the curves and tangent line. First, let’s try \texttt{help textplot} to learn how to use this command in Maple.

\begin{verbatim}
> ? textplot
> t1:=textplot([0.01,tanline(0)+0.01,‘tangent line’],align={ABOVE,RIGHT});
> display({p1,p2,t1});
> t2:=textplot([-0.7+0.01,f(-0.7)-0.01, ‘f(x)’],align={BELOW,RIGHT});
> display({p1,p2,t1,t2});
> t3:=textplot([0.2+0.01,D(f)(0.2)+0.01, ‘f’(x)’],align={ABOVE,RIGHT});
> display({p1,p2,t1,t2,t3});
\end{verbatim}

4. Exercises

- 1. Use Maple to find the derivatives of the following functions. In each case plot the function and its derivative on the same graph, for \( x = -2..2 \). Can you tell which curve is the original function and which is the derivative?
  
  a) \( x(x - 1)(x + 2) \);
  
  b) \( \frac{x^2-1}{\cos(\frac{x}{2})} \).

- 2. An alternative to the graphical approach in investigating slopes and derivatives is to make a table showing the slope of the secant line through \( (x_0, f(x_0)) \)
and \((x_0 + h, f(x_0 + h))\) for various values of \(h\) approaching 0. Try this, using the \texttt{chart} command, for the following functions and points. Be sure to take both positive and negative \(h\) values, and watch out for roundoff error that could cause difficulty if \(h\) is extremely small.

a) \(\sin(x), x = \frac{\pi}{4}\);  
b) \(\frac{1}{\sqrt{x^2 + 1}}, x = 1\);  
c) \(\sqrt{1 - \sin(x)}, x = \pi\);  
d) \(\text{sgn}(x) \sqrt{1 - \sin(x)}, x = \pi\).

3. Investigate graphically the secant lines to the graphs of the following functions through the given points. Does a tangent line exist? If so, what is it? If Maple doesn’t find the slope successfully using \texttt{D}, you should try taking the left and right limits of the \textbf{Newton quotient} \(\frac{f(x_0 + h) - f(x_0)}{h}\).

a) \(\frac{x+1}{x^2+1}\), point (1,1);  
b) \(|x^2 - 1|\), point (1,0);  
c) \(-\sqrt{|x|}\) if \(x < 0\), \(\sqrt{x}\) otherwise, point (0,0);  
d) \(x^2 \sin\left(\frac{1}{x}\right)\) if \(x\) is not zero, 0 if \(x = 0\), point (0,0);  
e) \(x^2\) if \(x \leq \frac{1}{2}\), \(x - \frac{1}{4}\) otherwise, point \(\left(\frac{1}{2}, \frac{1}{4}\right)\).

4. Consider the curve \(y = \frac{x^2}{x-1}\). Where does it have horizontal tangents? Plot these together with the curve. Do the same for the tangents of slope -1. Use reasonable \(x\) and \(y\) intervals so that the points of tangency are visible.

5. a) What are all the possible slopes of tangent lines to the curve \(y = \frac{x^2}{x-1}\)? \textit{Hint:} solve the equation \(y' = m\).  
b) For what values of \(m\) is there exactly one point on the curve where the tangent line has slope \(m\)?  
c) What are all the possible slopes of secant lines for this curve?

---

**Lab 15. Derivatives in Maple and Differentiation Formulas**
In this Lab we introduce two different differentiation operators. You will also discover some useful algebraic patterns occurring when computing derivatives. Once we have identified and confirmed these patterns, they remain available to help us streamline all of our computations – both by hand and by machine.

1. “diff” and “D”
There are actually two differentiation operators in Maple: diff, which is most useful for expressions, and D, which is most useful for functions.
If \( f \) is a function of one variable, \( D(f) \) is its derivative. This is also a function of one variable. As with any function, you evaluate it at some \( p \) by putting \( p \) in parentheses after the function: \( D(f)(p) \), with two sets of parentheses.

\[
\begin{align*}
> & f := x \rightarrow x^2 + a \cdot x; \\
> & D(f); \\
> & D(f)(p);
\end{align*}
\]
If Maple doesn’t know how to differentiate a function (in particular, if that function hasn’t been defined), it will leave \( D \) of that function as it is.

\[
\begin{align*}
> & D(f\text{new})(p);
\end{align*}
\]
If expr is an expression, \( \text{diff(expr,x)} \) is its derivative, where \( x \) is the independent variable. This is also an expression. The variable \( x \) must not be assigned a value when you use \( \text{diff} \), or that value would be used instead of \( x \) and would probably cause an error. You can use \( \text{subs} \), or assign the derivative to a name before assigning a value to \( x \).

\[
\begin{align*}
> & \text{expr} := x^2 + a \cdot x; \\
> & v := \text{diff(expr,x)}; \\
> & \text{subs}(x = 2, v); \\
> & x := 2; \\
> & \text{diff(expr,x)};
\end{align*}
\]
(assigning a value to \( x \) caused this error: Error, wrong number (or type) of parameters in function \( \text{diff} \)).

\[
\begin{align*}
> & v;
\end{align*}
\]
In the DOS, an unevaluated \( \text{diff(expr,x)} \) is shown in Maple’s output as \( \frac{d}{ds} \text{expr} \). In the Windows version, it is \( \frac{\partial}{\partial x} \text{expr} \). You will meet partial again in calculus of several variables; for now, you can think of it as being the same as \( d \).
Any variable, other than the independent variable, that hasn’t been given a value is considered to be a constant when differentiating. An example is \( a \) in \( D(f) \) and \( \text{diff(expr,x)} \) above. In \( \text{diff(expr,x)} \), Maple uses the values of any variables that occur in \( \text{expr} \). So if \( a \) had the value \( 2 \cdot x \) in the example above, \( \text{expr} \) would have been taken as \( 3 \cdot x^2 \) and \( \text{diff(expr,x)} \) would have been calculated as \( 6 \cdot x \).
On the other hand, \( D(f) \) does not use the values of variables. It considered any name, other than the independent variable, occurring in the definition of \( f \) as a constant. So \( D(f) \) would be \( x \rightarrow 2 \cdot x + a \) regardless of the value of \( a \); if \( a \) was \( 2 \cdot x \) then \( D(f)(x) \)
would be calculated as $4x$. Therefore, when using $D$ you should be careful to avoid using any variables defined as expressions involving the independent variable. $D$ also does not work with functions defined in terms of other user-defined functions. Thus, in trying to avoid the problems of the last paragraph, you might be tempted to try

```maple
> a:=x->2*x;
> f:=x->x^2+a(x):
> D(f);
```

Maple just refuses to differentiate this function! A way to work around this is to use `unapply` to define the function $f$, so that the definition does not actually contain $a$.

```maple
> f:=unapply(x^2+a(x),x);
> D(f);
```

Since differentiating a formula is a purely mechanical calculation, it is not surprising that Maple can calculate the derivative of just about any function you might write down. It uses essentially the same rules that you learn in calculus (product rule, chain rule, etc.). There are just a few surprises.

```maple
> D(abs);
```

It looks a bit strange, but it has the right values: 1 if $a$ is positive, -1 if $a$ is negative, and undefined if $a$ is 0. In fact:

```maple
> D(abs)(0);
```

Similarly,

```maple
> D(signum);
```

where `signum(1,a)` is 0 if $a$ is not zero and returns an error if $a = 0$. On the other hand,

```maple
> D(Heaviside);
```

You will probably meet the Dirac function (which is not really a function) if you study Laplace transforms or Fourier transforms. Engineers and physicists use it frequently. This is not the proper place to explain it: I’ll just mention that $\text{Dirac}(x)$ is 0 if $x$ is not zero and does not have a numerical value if $x = 0$.

For a function defined piecewise, you can’t necessarily trust $D$ or `diff` at the endpoints of the intervals. In those cases, you should resort to the definition of derivative as a limit. For example, consider the function $f(x) = 0$ if $x < 0$, $f(x) = x$ otherwise.

```maple
> f:=x->piecewise(x<0,0,x);
> D(f)(0);
```

you’ll get the wrong answer!

```maple
> limit((f(h)-f(0))/h,h=0);
```

this will give you the right answer!

Let’s look at another example which has the derivative value at the endpoint.

```maple
> g:=x->x^2*Heaviside(x);
> limit((g(h)-g(0))/h,h=0);
```
2. Exceptional Points
We often must deal with a function defined by a certain expression except for one or two points, where the original expression may be undefined but a certain value is specified, e.g. \( f(x) = \frac{1}{x} \) if \( x \) is not zero, \( f(x) = 0 \) if \( x = 0 \).
The simplest way to deal with this in Maple is to define \( f \) using the expression, and then define the value of \( f \) at the special points:
\[
\begin{align*}
\texttt{f:=x->1/x;} & \\
\texttt{f(0):=0;} & \\
\end{align*}
\]
Be sure to enter the \( f:=... \) before the special values, as \( f:= \) will wipe out any special values defined previously.

3. Differentiation Formulas
During our search for formulas we will frequently need to refer to the quotient
\[
\frac{f(x+h)-f(x)}{h};
\]
The following Maple procedure can be used to generate such a quotient for a given function \( f \).
\[
\begin{align*}
\texttt{with(student):} & \\
\texttt{nq:=makeproc(%%,f);} & \\
\end{align*}
\]
Here are some examples of its use.
\[
\begin{align*}
\texttt{nq(f);} & \\
\texttt{nq(g);} & \\
\texttt{nq(f+g);} & \\
\end{align*}
\]
With these, we can work out some differentiation formulas by ourselves.

Constant functions: if \( f \) is a constant function so that \( f(x) = c \) for some constant \( c \), then \( f' = D(f) \) is defined by \( D(f)(x) = 0 \) for all \( x \).
\[
\begin{align*}
\texttt{f:=x->c;} & \\
\texttt{D(f);} & \\
\end{align*}
\]

Pure powers: the derivative of the function \( f \) defined by \( f(x) = x^\alpha \) where \( \alpha \) is a real number, is the function \( f'(x) = \alpha x^{(\alpha-1)} \) for all \( x \).
\[
\begin{align*}
\texttt{f:=x->x^alpha;} & \\
\texttt{D(f);} & \\
\end{align*}
\]

Derivatives of combinations of expressions can be computed by making use of the following basic rules.
\[
\begin{align*}
\texttt{constants:=constants,c: f:='f';} & \\
\texttt{D(c*f);} & \\
\texttt{D(f+g);} & \\
\texttt{D(f*g);} & \\
\end{align*}
\]
Note: the equivalent rules for expressions in $x$ are
\[
\text{diff}(c\cdot f(x), x);
\]
\[
\text{diff}(f(x)+g(x), x);
\]
\[
\text{diff}(f(x)\cdot g(x), x);
\]
The derivative of the function $v$ defined by $v = \frac{1}{g}$ is
\[
v:= \frac{1}{g};
\]
\[
D(v);
\]
When this is evaluated at $x$, we obtain the expression
\[
D(v)(x);
\]
which can also be written as
\[
diff(\frac{1}{g(x)}, x);
\]
By using the product rule on $\frac{f}{g}$, we obtain the following result.
\[
v:= \frac{f}{g};
\]
\[
D(v);
\]
When this function is evaluated at $x$, we obtain the expression
\[
D(v)(x);
\]
which can also be written as
\[
diff(v(x), x);
\]
The rules for sums and products allow us to combine these to produce formulas for an even wider class of derivatives.
\[
f:= x\rightarrow x^2\cdot \sqrt{x-7};
\]
\[
f(x), D(f)(x);
\]
\[
a:= \text{randpoly}(x, \text{degree}=3);
\]
\[
b:= \text{randpoly}(x, \text{degree}=4);
\]
The quotient of polynomial is
\[
a/b;
\]
By the product rule, the derivative is
\[
diff(a, x)/b+a*diff(1/b, x);
\]
which is the same as
\[
diff(a/b, x);
\]
The more traditional formula for quotients gives the result in the form
\[
\text{normal}(\%);
\]
Derivatives of trigonometric functions
\[
f:= x\rightarrow \sin(x); \quad f(x), D(f)(x);
\]
\[
f:= x\rightarrow \cos(x); \quad f(x), D(f)(x);
\]
\[
f:= x\rightarrow \tan(x); \quad f(x), D(f)(x);
\]
\[
f:= x\rightarrow \cot(x); \quad f(x), D(f)(x);
\]
\[
f:= x\rightarrow \sec(x); \quad f(x), D(f)(x);
\]
\[
f:= x\rightarrow \csc(x); \quad f(x), D(f)(x);
\]
Also we can work out the derivatives of other elementary functions: \( f(x) = a^x, \)
\( f(x) = \log_a(x), \)
\( f(x) = e^x, \)
\( f(x) = \ln(x), \)
\( f(x) = \arcsin(x), \)
\( f(x) = \arccos(x), \)
\( f(x) = \arctan(x), \)
\( f(x) = \arccot(x), \)
etc.

Another important way of constructing new functions from old is through functional composition. Specifically, we need to be able to deal with expressions of the form \( f(g(x)) \) (the chain rule). If \( f \) and \( g \) are differentiable functions with domains defined in such a way that \( v = f \circ g \) is defined on a domain \( S \), then
\[
> v := 'v' : f := 'f' : g := 'g' : D(v) = D(f@g);
\]
The result of applying the function \( D(v) \) to \( x \) is
\[
> D(v)(x) = D(f@g)(x);
\]

### 4. Exercises

- **1.** Where is the function \( f(x) = |x^3 - x| - 2 |x - 1| \) non-differentiable? Where does it have a horizontal tangent? Where does it have a tangent of slope 2?

- **2.** To see that Maple knows the differentiation rule for sums, you could enter
\[
> D(f+g)(x);
\]
where \( f, g \) and \( x \) have not been given values. How does Maple express the Product Rule, Reciprocal Rule, Quotient Rule and Chain Rule?

- **3.** Consider the curve \( y = \frac{x^2}{x^2-1} \).
  a) At what other point(s) does a tangent line at \( (x_1, y_1) \) intersects the curve? 
     *Hint:* \( f(x_2) = f(x_1) + D(f)(x_1)(x_1 - x_2) \). Discard the case \( x_2 = x_1 \).
  b) Can the same line be tangent to this curve at two or more different points? 
     *Hint:* this would mean \( D(f)(x_2) = D(f)(x_1) \) in addition to the condition of part (a).

- **4.** For rational functions (i.e. quotients of polynomials), you can greatly reduce difficulties with roundoff error in computing difference quotients if you use the normal command on the expression before evaluating it at a particular \( h \). This essentially performs the same trick we use in calculating a derivative “from the definition”, simplifying the difference quotient and canceling a factor of \( h \) from the numerator and denominator. Try this for \( f(x) = \frac{x^3-x}{x^2+x+1} \) at \( x_0 = 2 \). Use chart to produce a table showing the difference quotient and the normal version of the difference quotient for various small values of \( h \). Include values small enough that the difference quotient (in its original form) evaluates to 0.
5. Consider the function \( f(x) = \frac{1}{x^2 - 2x + 2} \). Find the difference, quotient and derivative of this function at a general point \( x \). Use this to evaluate the derivative at the points \( x = -1 \) and \( x = 4 \). Plot \( f \) and its derivative on the same set of axes, using an appropriate domain. At what value of \( x \) is the derivative equal to zero? For what value of \( x \) does the tangent line to \( f \) appear to have zero slope?

6. Define the function \( f(x) = \frac{|1-x^2|}{1-x} \). Plot it and its derivative on the same set of axes. From the graph, read off the slope of the tangent line to the graph of \( f(x) \) as you approach \( x = 1 \), first approaching from the left and then from the right. Can you define a tangent line exactly at \( x = 1 \)? Using the graph of the derivative of \( f(x) \), decide what are the limits of this derivative as you approach \( x = 1 \) from the left and from the right. Does the derivative have a value at \( x = 1 \)?

7. Plot \( y = x^4 - 3x^3 + \frac{x}{2} - 3 \). Where does the tangent line have slope equal to zero? It is rather hard to tell by inspection, so instead use the following method: First compute the derivative of \( y \) and call the resulting expression \( y' \). We will apply a new command, \texttt{fsolve( )} to this derivative. You can apply the \texttt{solve( )} command to the derivative of \( y \), but the response may not be very helpful. Now plot both \( y \) and its derivative on a single set of axes (be careful to choose a useful domain and range). Which is which? Use information from the graph of the derivative to indicate those points where the tangent to the graph of \( y \) has zero slope.

8. Find the derivative of \( \sqrt{x^2 + 3} \), which is the composition of the functions \( g(x) = \sqrt{x} \) and \( h(x) = x^2 + 3 \), using the chain rule.
   \[
   > \quad g:=x->\text{sqrt}(x): \quad h:=x->x^2+3: \quad (g@h)(x); \\
   \]

9. Find \( \frac{du}{dx} \) when \( y = u^3 + 3u^2 + 3u + 1 \) and \( u = x^2 + 3 \). What is its value at \( x = 3 \)?

10. Use the definition of the derivative (define the difference quotients, then find the limit) to compute
    a) \( f'(2) \), where \( f(x) = \sqrt{4x+1} \); b) \( f'(0^+) \), where \( f(x) = x\sqrt{4x-x^2} \).
11. Use the properties of derivatives and the formulas for differentiation to find derivatives of the following functions
   a) \( g(x) = \arctan(x - \sqrt{1 + x^2}) \);
   b) \( h(x) = \ln(\arctan(\sqrt{1 + x^2})) \);
   c) \( f(u) = \arcsin(\frac{2}{u}) \);
   d) \( y(x) = \left(\frac{1 + \tanh(x)}{1 - \tanh(x)}\right)^{\frac{1}{4}} \).

12. Check if the function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x \arctan(\frac{1}{x}) \) for nonzero \( x \) and \( f(x) = 0 \) for \( x = 0 \), has a derivative at \( x = 0 \). If \( f'(0) \) does not exist, check if \( f'(0^-) \) and/or \( f'(0^+) \) exist.

13. Verify if the function \( f(x) = \sqrt{\ln(1 + x^2)} \) is differentiable at \( x = 0 \).

Lab 16. Differentials
One of the most important uses of tangent lines is to construct approximations. If the graph of a function is sufficiently smooth, we can estimate values of the function near a specific point on the curve by assuming the curve is identical to the tangent line at that point. The resulting formula is usually much easier to compute and, providing that we remain close to the point where the tangent line actually meets the curve, the results can be quite accurate.

1. Theory and Some Examples
   The slope of a line tangent at \( x = x_0 \) to the curve defined by \( f \) is
   \[
   \frac{d}{dx} f(x_0) = \frac{dy}{dx} \mid _{x=x_0} = \frac{d}{dx} \left( \sqrt{\ln(1 + x^2)} \right) \mid _{x=x_0} = \frac{1}{2} \cdot \frac{1}{\sqrt{\ln(1 + x^2)}} \cdot \frac{2x}{1 + x^2} = \frac{x}{\sqrt{\ln(1 + x^2)}}.
   \]
   and so the equation for the line tangent at the point \((x_0, f(x_0))\) is
   \[
   eq1 := y - f(x_0) = D(f)(x_0) \cdot (x - x_0);
   \]
   which can also be rearranged as
   \[
   eq2 := isolate(eq1, y);
   \]
   For the details of the command \texttt{isolate()}, please try \texttt{?isolate}. Remember execute the command \texttt{readlib(isolate)} before you execute the command \texttt{isolate}.

   The quantity \( y - f(x_0) \) is \( eq1 \) is an estimate of the change in the function value generated by moving from \( x_0 \) to \( x \) and is usually denoted by \( dy \). The exact change in the \( x \) value is denoted by \( dx \) so the \( eq1 \) can be written as \( dy = D(f)(x_0) \cdot dx \).
   Notice that \( dy \) is only an estimate of the change in function value and that \( f(x_0) + dy \) is only an estimate of the value \( y = f(x) \).

   Let’s take a look at several examples. Consider the function \( f(x) = x^3 + 2x^2 \). If we want to get the approximate values for the points near \( x = 1 \), we can use the tangent line at \( x = 1 \).
The slope of the tangent line at \(x = 1\) is
\[
D(f)(1);
\]
The tangent line
\[
y - f(1) = D(f)(1)(x - 1);
\]
approximates the function \(f\) near \(x = 1\) so that, for example,
\[
isolate(\%, y);
\]
is an approximation to \(f(x)\). The function \(f\) and its tangent line appear in the graph
\[
\text{with(student): showtangent}(f(x), x=1, x=0..2);
\]
The change in the function value caused by moving from \(x = 1\) to \(x = a\) is approximated by
\[
D(f)(1)(a - 1);
\]
This approximation can be quite accurate if we remain close to the point \(x = 1\), i.e. \(a\) is very near 1, such as \(a = .9\), \(a = 1.01\), etc. For example, the exact value and the approximate value at \(x = .9\):
\[
> f(0.9), f(1)+D(f)(1)*(0.9-1);
\]
The exact value and the approximate value at \(x = 1.01\):
\[
> f(1.01), f(1)+D(f)(1)*(1.01-1);
\]
The results are very good. But if we take \(a = 2\), then the approximation is far away from the exact value.
\[
> f(2), f(1)+D(f)(1)*(2-1);
\]
This kind of approximation could be very useful sometimes, especially for the case when the function value is hard to calculate at an arbitrary point \(x = a\), but while the nearby of the point \(x = x_0\) this value can be easy to estimate. Furthermore, one of the greatest achievement from this approximation is **Newton’s Method**, one of the very useful numerical method for solving equations.

Consider the function \(f(x) = (\frac{1}{1+x^2})^{\frac{1}{3}}\), we know \(f(0) = 1\), it’s very easy for us to get an approximation of \(f(.01)\) by using the differentials.
\[
> f:=x->\text{root}(1/(1+x^2)^3,3);
> \text{diff}(f(x),x);
> m:=\text{subs}(x=0,\%);
> f(0.01), f(0)+m*(0.01-0);
\]

2. Derivation of Newton’s Formula
Since \(f(x_0 + h)\) can be approximated by \(f(x_0) + hD(f)(x_0)\). We can find \(h\) (and hence \(x_0 + h\)) such that \(f(x_0 + h) = 0\). We simply solve the equation \(f(x_0) + D(f)(x_0)h = 0\) as
\[
> f:='f':
> \text{solve}(f(x0)+D(f)(x0)*h=0,h);
\]
The new \(x\) value is \(x_1\) given by
and it is generally a better guess at the solution to \( f(x) = 0 \) then \( x_0 \).
By using this formula over and over again with better and better guesses for \( x_0 \) we should be able to guess the value of the root of the equation as accurate as we want to.

The following Maple procedure takes a function \( f \) and an initial guess start value and generates new guesses until the value of the function \( f \) at the current guess is sufficiently small.

```maple
> newton:=proc(f,startvalue) local x0,x1;
> x0:=startvalue;
> while abs(f(x0))>0.001 do
> x1:=x0-f(x0)/D(f)(x0);
> x0:=x1;
> print('current approximation: ',x0);
> od;
> RETURN(x0);
> end:
```

For example, find an approximate solution to \( \sin(x^2 - 3) = 0 \).

```maple
> f:=x->sin(x^2-3);
> newton(f,1.0);
> newton(f,5.0);
> newton(f,10.0);
```

Depending on where we start, we obtain different roots.

```maple
> plot(f,0..10);
```

The exact solutions for the equation \( \sin(x^2 - 3) = 0 \) are \( x_n = \sqrt{n \pi + 3} \) and \( y_n = -\sqrt{n \pi + 3} \), where \( n \) could be any integer that makes sense.

```maple
> evalf(-sqrt(3),10);
> evalf(sqrt(7*Pi+3),10);
> evalf(sqrt(31*Pi+3),10);
```

3. Exercises

- 1. Approximate \( f(a) \) when \( a \) is near \( x = x_0 \), by using the tangent line at \( x = 0 \), for the following functions
  a) \( f(x) = (1 + x^2)^{\frac{3}{4}} \), \( x_0 = 0 \);
  b) \( g(x) = x \arcsin(x) + \sqrt{1 - x^2} \), \( x_0 = 0 \);
  c) \( f(x) = \sqrt{4x + 1} \), \( x_0 = 2 \);
d) \( f(x) = x \sqrt{4x - x^2} \), \( x_0 = 2 \);

- 2. Find the error incurred by using the tangent line at \( x = 2 \) to estimate \( f(2.5) \) if the function \( f \) is defined by \( f(x) = x^3 + 3x^2 - 2x + 1 \).

- 3. Use the tangent line at \( x = 3 \) to approximate the change \( dy \) as an expression in \( x \) if \( f(x) = \frac{\sin(3x^2)}{x} \). Use this to estimate \( f(3.5) \).

- 4. Find the exact error incurred by using the differentials at \( x = 2 \) to estimate \( f(2.01) \) if the function is \( f(x) = x^3 + 4x^2 - 6x + 2 \).

- 5. Use Newton’s algorithm to find a solution to the equation \( x^3 - 4 = 0 \). Compare your result with that obtained by \texttt{fsolve}() .

- 6. Use Newton’s algorithm to find at least three solutions to the equation \( \cos(3x^2 - 5) = 0 \). Compare your results with that obtained by \texttt{fsolve}() . You can use the \texttt{plot()} to help identify appropriate intervals and starting points.

**Lab 17. Fundamental Properties of Differentiable Functions**

First we will review several theorems concerning the fundamental properties of differentiable functions. Then explain them with the help of Maple and graphs. This Lab will give you an intuitive understanding of these fundamental properties. In this Lab, we assume that all discussed functions are differentiable, and \( x_0 \) in \( \text{Dom}(f) \) is an accumulation point of \( \text{Dom}(f) \).

1. **Theorems and Their Applications**

   \( f \) has a local extremum if it has either a local minimum (there exists a neighborhood of \( x_0 \), such that all the values of \( f \) in this neighborhood are greater than \( f(x_0) \)) or a local maximum (there exists a neighborhood of \( x_0 \), such that all the values of \( f \) in this neighborhood are less than \( f(x_0) \)). On the graph, the local minimum will be shown as bottom point, the local maximum will be shown as top point, while they
are all local. For the following graph, can you recognize that \( x = -4, x = 1, x = 4 \)
are top points and \( x = -1, x = 2, x = 5 \) are bottom points?

![Graph of a function with critical points identified](image)

Fermat’s theorem gives you a method of finding a local extrema of differentiable functions numerically: If \( f \) has a local extremum at \( x_0 \) and \( D(f)(x_0) \) exists, then 
\( D(f)(x_0) = 0 \). From this, we can first solve the equation of \( D(f)(x) = 0 \) for \( x \). 
The roots are possible local extrema. Also, the undefined \( D(f)(x) \) are possible local extrema. Later on, you will learn that these points are called critical points. 
Once we get the critical points, we compute some values near those points to see if they are local maxima or local minima.

```maple
> f:=x->2*x^3-15*x^2+36*x+70;
> D(f)(x);
> solve(%,x);

D(f)(x) is well defined on the real line. There are two possible local extrema: \( x = 2 \)
and \( x = 3 \), which divide the real line into three intervals. Compute the values of \( f(2) \), 
\( f(3) \), and the values of this function at one point in each interval.

```maple
> f(0),f(2),f(2.5),f(3),f(4);
```

We see that \( x = 2 \) is a local maximum since \( f(0) < f(2) \) and \( f(2.5) < f(2) \), \( x = 3 \) is 
a local minimum since \( f(3) < f(2.5) \) and \( f(3) < f(4) \). 
For the function \( f(x) = x^3 \), there is a little bit different.

```maple
> f:=x->x^3;
> D(f)(x);
> solve(%,x);
> f(-0.001),f(0),f(0.001);
```

From the above calculation, \( x = 0 \) is the only possible local extremum. but \( f(-.001) < f(0) < f(.001) \). Thus, \( x = 0 \) is neither a local maximum nor a local minimum.


Darboux theorem is the intermediate value property for the derivative, where there is no need to assume that the derivative is continuous. However, we ought to remember the condition: \( D(f)(x) \) exist for all \( x \) in \([a,b]\). For example, \( f(x) = |x| \) for \( x \) in \([-2,2]\). \( D(f)(0) \) doesn’t exist, thus the theorem can not be applied. Consider the previous function \( f(x) = 2x^3 - 15x^2 + 30x + 70 \) for \( x \) in \([0,5]\).

\[
> D(f)(0), D(f)(5); \\
> \text{solve}(D(f)(x)=50, x); \\
> \text{evalf}(\%);
\]

\( D(f)(0) = 0 \) and \( D(f)(5) = 75 \). Pick up any number \( \alpha \) between 0 and 75, there exists at least one \( x_0 \) in \([0,5]\), such that \( D(f)(x_0) = \alpha \). In the above example, \( \alpha = 50 \) and \( x_0 = 4.082482906 \).

Rolle’s theorem states that for a differentiable function, if \( f(a) = f(b) \) there is at least one horizontal tangent line to be found somewhere between \( a \) and \( b \).

\[
> f := x -> -x^2 + 8x + 288; \\
> f(2), f(6); \\
> D(f)(4);
\]

Lagrange theorem is a generalization of Rolle’s theorem. If \( f(a) = f(b) \), the line connecting these two points \((a, f(a)) \) and \((b, f(b))\) is a horizontal line, the slope is 0. What if \( f(a) \) is not equal to \( f(b) \)? Then this line is not horizontal, and the slope is not equal to 0. Is there anything special about this general case? The slope of the line connecting \((a, f(a)) \) and \((b, f(b))\) is \( \frac{f(b)-f(a)}{b-a} \) is called the average slope. By Lagrange theorem, there is at least one tangent line with the same slope as the average slope to be found somewhere \( x_0 \) between \( a \) and \( b \).

Let’s consider the function \( f(x) = 3x^4 - 16x^3 + 18x^2 \) for \( x \) in \([-1,3]\).

\[
> f := x -> 3*x^4 - 16*x^3 + 18*x^2; \\
> A := [-1, f(-1)]; B := [3, f(3)]; \\
> \text{two endpoint of the curve over the interval \([-1,3]\)} \\
> \text{with(student):} \\
> g := \text{makeproc}(A, B); \\
> \text{the procedure to draw the line connecting two end points} \\
> \text{plot}\{\{f, g\}, -1..3\}; \\
> \text{plot the two functions} \\
> f \text{ and} \\
> g \text{ on the same axes} \\
> \text{slope}(A, B); \\
> \text{get the slope of the line AB} \\
> \text{st := fsolve(D(f)(x) = -16, x)}; \\
> \text{find the same tangent slopes}
\]

The answer shows three solutions, which means that there are three tangent lines having the same slope as the average slope. We will put these graphs together.

\[
> t1 := x -> -16*(x-st[1])+f(st[1]);
\]
Appendix 2: Introduction to Maple V Labs

> t2:=x->-16*(x-st[2])+f(st[2]);
> t3:=x->-16*(x-st[3])+f(st[3]);
> plot[display]({t1,t2,t3,f,g},-1..3);

The graph confirms that the tangent lines and the line depicting the average slope are parallel.

2. Exercises

- 1. Use Fermat’s theorem to find local extrema of the function given by \( f(x) = (7 + x) (11 - 3x)^{\frac{3}{2}} \). Plot the graph of this function together with the tangent line at the local extrema.

- 2. Find the local extrema of the function \( f(x) = \sin(2x) \). Show that the maxima is achieved at more than one location. Use graphics and \texttt{fsolve} to find approximate locations for several of these extrema.

- 3. Consider the function \( f(x) = x^2 + 2 \). Investigate the position and value of the extrema points of the function \( f(x + 2) \), \( f(x - 2) \), \( f(3x) \), and \( f(-x) \). How are these related to the extrema point(s) of \( f(x) \)?

- 4. Find the local extrema of \( f(x) = x\,|2 - x|^{\frac{5}{4}} \).

- 5. Find the local extrema of \( f(x) = |x| \) on the interval \([-2,2]\).

- 6. We can use Rolle’s theorem to find a local extremum for a function \( f \), but we require \( a \) and \( b \) for which \( f(a) = f(b) \). Use \texttt{solve} or \texttt{fsolve} to find suitable values for \( a \) and \( b \) for the function defined by \( f(x) = -t^2 + 4t + 300 \).

- 7. Consider \( f(x) = x^3 - 6x^2 + 13x - 11 \). How many solutions to \( f(x) = 0 \) are there? Locate any roots you find more precisely by plotting \( f \) over a more restricted domain. Examine the graph of \( D(f) \). Use this in conjunction with Rolle’s theorem to prove that there is only one real solution to \( f(x) = 0 \).

- 8. Plot the function \( f(x) = x^3 - 3x^2 + 2x \) on the interval \([0,3]\). Find the point \( c \) in \((0,3)\) such that \( D(f)(c) \) is equal to the slope of the line connecting the
two points \([0, f(0)]\) and \([3, f(3)]\). Plot the graph of function \(f(x)\), the line connecting two endpoints \([0, f(0)]\) and \([3, f(3)]\), and the tangent line at \(c\) on the same axes. Does the tangent line at \(c\) parallel to the line connecting two endpoints?

**Lab 18. Applied Maximum and Minimum Problems**

Many practical problems involve maximizing or minimizing some quantity. If we can model these problem by some differentiable function, then we can use the techniques explained in the textbook to find the values of \(x\) that maximizes or minimizes this function \(f(x)\).

1. **Finding Maxima/Minima of a Function**

A function has an absolute maximum value at \(c\) if \(f(x) \leq f(c)\) for all \(x\) in the domain of \(f\). A function has an absolute minimum value at \(c\) if \(f(x) \leq f(c)\) for all \(x\) in the domain of \(f\).

In order to find maximum and minimum value of a function \(f\), we first need to determine the domain \(\text{Dom}(f)\) of the function \(f\), preferably as a closed interval, then solve the equation \(D(f)(x) = 0\) for \(x\) to find all the critical points \(\{x \in \text{Dom}(f) | D(f)(x) = 0\} \) or \(D(f)(x)\) undefined, then compare the values of the function \(f\) at these critical points and at the endpoints. The largest of these values is the maximum of \(f\), while the smallest value is the minimum.

For example, find the absolute minimum and the absolute maximum of the function \(f(x) = x^2 + 2\) on the interval \([-2, 4]\).

First, we notice that we only need to consider the function \(f\) on the interval \([-2, 4]\). There are two endpoints \(x = -2\) and \(x = 4\). Solve the equation \(D(f)(x) = 2x = 0\), we get \(x = 0\) only one solution. There is only one critical point \(c = 0\). Compare \(f(0) = 2\), \(f(-2) = 6\), and \(f(4) = 18\), the absolute minimum is \(f(0) = 2\) and the absolute maximum value is \(f(4) = 18\).

Another example, find the absolute maximum and absolute minimum for the function \(f(x) = \frac{\sin(x) + x}{1+x^2}\).

\[
> f:=x->(\text{abs}(\sin(x)) + x)/(1+x^2);
\]

\[
> \text{plot}(f(x), x=-\text{infinity}..\text{infinity});
\]

From the most general graph of this function, we know that there are finite absolute maximum and absolute minimum. To find the absolute maximum, let’s get a closer look at the peak point.

\[
> \text{plot}(f(x), x=-1..5);
\]

When you double click at the peak point, you may guess that \(x\) is in \([0.7, 1.1]\). There should be a critical point for \(x\) in this interval. Plotting out the derivative of this function on this interval confirms that there is a critical point \(c\) such that \(D(f)(c) = 0\).
Appendix 2: Introduction to Maple V Labs

> plot(D(f)(x), x=0.7..1.1);
> fsolve(D(f)(x)=0, x, x=0.7..1.1);
> f(%);

Use the numerical solver fsolve to precisely locate the critical point. Combine with the graph of the function, we know that the maximum value, .9303082089, occurs at 
\(x = .8799272979\). Next we look for the minimum.

> plot(f(x), x=-4..-2, discont=true);
> plot(D(f)(x), x=-3.3..-2.9, discont=true);

We see that there is a jump near \(x = 3.14\), which means \(D(f)(x)\) is undefined at this point. From the expression of the first derivative, the exact point is \(x = \pi\). This point is also a critical point.

> D(f)(x);
> D(f)(Pi);
> f(Pi);
> evalf(%);

Combine with the graph of the function, we see that the minimum value, .2890254823, occurs at \(x = \pi\).

2. Practical Problems

Where possible, we use plotting to understand better the behavior of the function and to estimate location where the extrema occur. The graphical representation provides intuition and approximations while the algebraic approach quantifies the relationships in a way that allow us to reuse them in related problems.

**Example 1.** A farmer has 2400 feet of fencing and wants to fence off a rectangular field bordering a straight river. He needs no fencing along the river. What are the dimensions of the field with the largest possible area?

First, we specify some known relationships between the area \(A\) and the perimeter \(P\):

> e1:=A=x*y: e2:=P=2*x+y:

Then we use the second of these to obtain one expression for \(y\) in terms of \(x\).

> sol:=solve(e2,y);

Use this to eliminate \(y\) from the first equation.

> subs(y=sol,e1);

In this way, we get an expression that can be used to compute the area of the field as a function of \(x\).

> with(student): f:=makeproc(rhs(%%),x);

Concerning the above two commands, since `makeproc` comes with `package student`, before we execute this command, we must call the `package student` by `with(student)`; for the details of using `makeproc`, please try `? makeproc`; the first argument of `makeproc` should be an expression, we put it as `rhs(%%)`, do you know why the `%%`?
For the real problem, $x$ (length of the fence) can not be negative or longer than $\frac{P}{2}$ (since $y$ cannot be negative). Thus the finite domain of the function $f$ is $[0, \frac{P}{2}]$. Now we need to maximize $f$ over the domain $[0, \frac{P}{2}]$. In the case of this example, $P = 2400$.

> P:=2400;
> plot(f,0..P/2);

From the graph, we find only one maximum. Use the algebraic method to find the exact point and value.

> solve(D(f)(x)=0,x);
> if this doesn’t work, try fsolve
> f(%);

Thus, the maximum value, 720000, occurs at $x = 600$. Hence the answer for the question is building the fence by 600 wide and 1200 long will give the largest possible area field.

**Example 2.** Find the area of the largest rectangle that can be inscribed in a semicircle of radius $r$.

Let’s first build the target function (express the area as the function of one variable). We use the fact: if one vertex of the inscribed rectangle is $(x, y)$ on the semicircle, then the area of the inscribed rectangle is $2xy$.

> e1:=x^2+y^2=r^2: A:=2*x*y:
> solve(e1,y);
> subs(y=%[1],A);
> choose the positive root for
> y
> A:=makeproc(%1,x);

Next we need to do is to maximize the function $A(x)$. For the real problem, the domain of this function is $[0,r]$. Similarly to the previous example, it’s not hard to find the maximum. We can not use the graph here since the perimeter $r$ is not specified.

> solve(D(A)(x)=0,x);
> c=%[1];
> choose the positive one while neglect the negative one
> A(c);

Hence, the largest area is $r \sqrt{r^2}$.

3. Exercises

- 1. A can is being designed to hold one litter of oil. Find the dimensions of the can that will minimize the amount of metal required to make the can.

- 2. Find the point on the parabola $y^2 = 3x$ that is closest to point (1,4).
• 3. A man is at a point A on a bank of a straight channel 3km wide. What is the fastest route to take to reach a point B, 8km down the far bank if the man can row at a speed of 6km/h and run at a speed of 8km/hr? Assume that his travel involves some rowing followed by running on the far shore and that there is no current involved.

• 4. A rectangular beam is to be cut from a circular log of radius 16in. If the strength of this beam is given by \((x, y) \rightarrow x y^2\) where \(x\) denotes the width and \(y\) denotes the depth, find the cross-sectional dimensions of the strongest beam that can be cut from this log.

• 5. Find the point on the parabola \(y^2 = 2x - 7\) that is closest to the point (2,7).

• 6. Let \(x\) denote distance down range, and \(y\) denotes the height. If the path of a projectile fired with an initial velocity \(v\) at an angle of inclination \(\theta\) above the horizontal can be approximated by the equation \(y = \tan(\theta) x - \frac{g x^2}{2 v^2 \cos(\theta)^2}\), where \(g = 9.8\text{m/sec}^2\) is the acceleration caused by gravity, find the angle \(\theta\) that maximizes the range if the terrain is level.

• 7. A 10cm by 100cm rectangular metal sheet is to be bent to form a 100cm long through, open at the top. The cross section of the through is to be a circular arc. How should this be done in order to maximize the volume of the through.

• 8. A triangle has one vertex at (0,0). The other two have the same \(y\) coordinate and are on the ellipse \(4x^2 + y^2 = 4\). What is the maximum area of such a triangle?

**Lab 19. Higher Derivatives and Taylor Formula**

In this Lab we will use Maple in calculating higher-order derivatives, including finding patterns for them. With the concept of higher-order derivatives, it won’t be hard to understand a very famous and very useful formula: Taylor Formula.

1. Higher Derivatives
Both of Maple’s differentiation operators D and diff can be used to calculate higher-order derivatives. \((D@@n)(f)\) is used to compute the \(n\)th derivative of a function \(f\). Remember to put parentheses around both the \(D@@n\) and the function name. If you are evaluating the function at \(x\), the \(x\) also goes in parentheses, so you have three sets of parentheses. \(\text{diff}(expr, x\$n)\) is used to compute the \(n\)th derivative of an expression \(expr\), where \(x\) is the independent variable.

\[
\begin{align*}
> & (D@@4)(f)(4); \\
> & \text{diff}(\text{expr}(x), x\$4); \\
> & (D@@1)(f)(x); \\
> & (D@@0)(f)(x); \\
> & \text{diff}(\text{expr}(x), x\$1), \text{diff}(\text{expr}(x), x); \\
> & \text{diff}(\text{expr}(x), x\$0);
\end{align*}
\]

\((D@@1)\) is the same as \(D\) and \((D@@0)(f)\) is \(f\). Similarly, \(\text{diff}(\text{expr}(x), x\$1)\) is the same as \(\text{diff}(\text{expr}(x), x)\). However, \(\text{diff}(\text{expr}(x), x\$0)\) does not work (it produces an error message).

The higher-order differentiable functions also have linearity property and Leibnitz formula:

\[
\begin{align*}
> & (D@@5)(f+g)(x); \\
> & (D@@4)(f*g)(x);
\end{align*}
\]

Maple won’t compute the general \(n\)th derivative for any function. If you specify the number \(n\), Maple will work the derivative for you. Let’s try to work out some examples from the text book.

\[
\begin{align*}
> & f:=x->x^\mu; \\
> & (D@@4)(f)(x); \\
> & f:=x->(a+b*x)^\mu; \\
> & (D@@3)(f)(x); \\
> & f:=x->a^x; \text{diff}(f(x), x\$5); \\
> & f:=x->\sin(x); (D@@6)(f)(x); \\
> & f:=x->\cos(x); (D@@7)(f)(x); \\
> & f:=x->\exp(x); (D@@20)(f)(x); \\
> & f:=x->\exp(a*x)*\sin(b*x); (D@@3)(f)(x); \\
> & f:=x->\arctan(x); (D@@1)(f)(x); (D@@2)(f)(x); (D@@3)(f)(x);
\end{align*}
\]

2. Taylor Formula

With the Taylor formula, we can approximate some functions by their Taylor polynomials. In this Lab, we will use Maple graphically and numerically to help you understand the error incurred in doing this Taylor polynomial approximation.

There are two Maple commands we need to use frequently in this section. First, \(\text{taylor}(expr, x=a, n+1)\) computes the degree \(n\) Taylor polynomial of \(expr\) about \(x = a\), while keeps the remainder term \(r_n(x)\) in the capital \(O\) form. \(\text{taylor}()\) doesn’t give
you exact Taylor polynomial, we need another command to cut off its last term $O(x^n)$. That’s the second command we will learn in this lab: \texttt{convert(expr,polynom)}, which converts the \texttt{expr} to a polynomial.

Let’s take an example for $f(x) = \sin(x)$. Compute a Taylor polynomial for this sine function to degree 4 about $x = 0$:

\begin{verbatim}
> t1:=taylor(sin(x),x=0,5);
\end{verbatim}

notice that we specify $n$ as 5 not 4.

Pay attention to the last term $O(x^5)$, which represents the remainder $r_4(x)$ (notation in the textbook). Maple uses this notation to tell us that $\sin(x)$ is equal to the polynomial we see displayed plus terms containing $x^5$, and possibly higher powers of $x$ as well. If we only want the Taylor polynomial not the extra remainder term $O(x^5)$, we need to convert the expression to a real polynomial.

\begin{verbatim}
> p1:=convert(t1,polynom);
\end{verbatim}

Next, we will compute Taylor polynomials of several different orders for the same function to compare the error incurred in so doing. To avoid typing \texttt{convert()} and \texttt{taylor()} command again and again, we first define a general function:

\begin{verbatim}
> tpoly:=n->convert(taylor(f(x),x=0,n+1), polynom);
\end{verbatim}

In this way, once we define the function $f(x)$, using \texttt{tpoly(n)} command we can find the Taylor polynomials for different choices of the order $n$ for the function $f(x)$.

\begin{verbatim}
> f:=x->sin(x);
> app1:=tpoly(6);
> app2:=tpoly(12);
> app3:=tpoly(20);
\end{verbatim}

Also it’s easy for us to evaluate the value of these polynomials, remember these polynomials are here expressions not functions.

\begin{verbatim}
> subs(x=0.5,app1);
\end{verbatim}

To translate the expression to a function, there is a way you may want to try. The following example shows how to change an expression \texttt{app1} to a function \texttt{fapp1}.

\begin{verbatim}
> fapp1:=subs(arg=x,body=app1,arg->body);
> fapp1(y);
> now fapp1 is a function
\end{verbatim}

Next, we plot out the graphs of these Taylor polynomials and the original function near $x = 0$:

\begin{verbatim}
> plot([f(x),app1,app2,app3],x=0..3*Pi, y=-2..2,color=[red,green, blue,yellow]);
\end{verbatim}

Which graph is for which function, expression? There are many ways to recognize the graph for the function. When plot the graphs, you can specify the colour option for each function. So that you can tell the function from the graphs by the colour you specified for that function. For the above example, we specified the function $f(x)$
as red colour, \texttt{app1} as green, \texttt{app2} as blue, and \texttt{app3} as yellow. For the details of this color option, please try \texttt{plot.color}.

From the graph, you see that four curves agree well for small \(x\), i.e. for \(x\) near \(x = 0\), but disagree when \(x\) is far away from \(x = 0\). Also, when you take a closer look, you will notice that after \(x = 2\), the green one (graph of tpoly(6)) goes away from the red one (graph of the original function), the blue one (graph of tpoly(12)) agrees with the red one (graph of sine function) very well before \(x = 4\), and the yellow one (graph of tpoly(20)) is almost the same as the red one before \(x = 7\). More Taylor polynomial terms you keep, more accurate approximation you will get.

Now, we will estimate the error numerically. Evaluate the Taylor polynomial and the original sine function at \(x = \frac{\pi}{3}\).

\begin{verbatim}
> subs(x=Pi/3,app1), subs(x=Pi/3,app2), subs(x=Pi/3,app3), f(Pi/3);
> evalf(%);
\end{verbatim}

Which one is the best approximation for \(\sin\left(\frac{\pi}{3}\right)\)? You can repeat this experiment, using different values \(x\), different orders of polynomials. What kind of result can you get from this graphs and numerical experiments? Does your result agree with the theoretical result in the text book?

3. Exercises

- 1. Let \(f(x) = x^n e^x\), find \((D^{(2)})(f)(x)\), \((D^{(4)})(f)(x)\), \((D^{(7)})(f)(x)\), and \((D^{(10)})(f)(x)\), can you get the general form for \((D^{(n)})(f)(x)\) using the Leibnitz property?

- 2. Consider the function \(f(x) = \frac{\ln(x)}{x}\), use Maple to compute the 3rd-, 5th-, 10th- and 17th-derivative of \(f(x)\). Check the results agree with the general expression of the \(n\)th-derivative \((D^{(n)})(f)(x)\) you worked out by using the Leibnitz property as the text book did.

- 3. \(f(x) = \cosh(ax)\), find \((D^{(1)})(f)(x)\), \((D^{(2)})(f)(x)\), \((D^{(3)})(f)(x)\), and \((D^{(4)})(f)(x)\). Can you guess the general form of the \(n\)th-derivative of this function?

- 4. Consider the function \(f(x) = \frac{1}{1-x}\), compute some Taylor polynomials of some orders (such as 5, 15, 100 degree) about \(x = 0\). Plot these graphs on the same axes, notice to choose the suitable interval of \(x\). Compare its exact value to the values of its Taylor polynomials you got at \(x = .85\), \(x = 1.1\). What do you notice happening? Do you have an explanation for it?
• 5. \( f(x) = \sin(x) \), compute the third-order Taylor polynomial at \( x = 0 \). Compare its exact value to the values of the Taylor polynomial at \( x = -0.3, x = 0.3, x = -0.2, x = 0.2, x = -0.4, \) and \( x = 0.4 \). Do these errors agree with the estimated error you deduced by the method discussed in the textbook?

• 6. \( f(x) = e^x \), compute the \( n \)th order Taylor polynomial at \( x = 0 \). Can you estimate the error? (\textit{Hint:} use the method discussed in the textbook). Choose several integer \( n \) (such as \( n = 5, 10, 20, \) etc.) and several value of \( x \) (such as \( x = 0.1, 0.5, 1, \) etc.) to verify your error estimation.

• 7. Use Taylor Theorem to approximate the value of \( \sqrt{e} \) with the accuracy .001. \textit{Hint:} Use Taylor polynomials for function \( f(x) = e^x \) at \( x = \frac{1}{2} \).

**Lab 20. Applications of First Derivatives**

The physical meaning of the first derivative can be interpreted as velocity, speed, rate of change, etc. It gives you the rate of change of the function value with respect to the variable. Here, we are talking about functions of one real variable. If the value of the function is growing, the rate will be positive, if the value of the function is decaying, the rate will be negative. Interesting thing is that the inverse way is also true provided some condition. Thus, we have the first derivative test.

1. First Derivative Test

There is some relationship between the graph of the function and that of its first derivative \( D(f) \). Take a look at this example \( f(x) = (x - 1)(x - 2)(x - 3)(x - 4) \).

\[
\begin{align*}
\text{f:=x->(x-1)*(x-2)*(x-3)*(x-4);} \\
\text{plot([f,D(f)],-1..5,-5..5,color=[red, green]);}
\end{align*}
\]

Notice that where \( f \) is increasing (left to right), the value of \( D(f)(a) \) is positive, where \( f \) is decreasing, the value of \( D(f)(a) \) is negative and where \( f \) has a local max or min, \( D(f)(a) \) is 0. This means that from the algebraic formula for a derivative, we can investigate the behavior of the original function. Without using any graphical tool, we can sketch the graph by the information we get from the first derivative formula. In the above example, \( f(1) = f(2) = f(3) = f(4) = 0 \), and there are three roots of the equation \( D(f)(x) = 0 \), which means that there are three possible local extrema.

\[
\begin{align*}
\text{solve(D(f)(x)=0,x);} \\
\text{evalf(\%);} \\
\end{align*}
\]
We know that points $I_1 = (1, 0), I_2 = (2, 0), I_3 = (3, 0), I_4 = (4, 0)$, $C_1 = (2.500000000, f(2.500000000)), C_2 = (3.618033989, f(3.618033989))$, and $C_3 = (1.381966011, f(1.381966011))$, where the points $I_1, I_2, I_3,$ and $I_4$ are the points when the graph cross the $x$-coordinate, while the points $C_1, C_2,$ and $C_3$ are either local max or min. All we need to do now is to associate these points to a curve. Since $D(f)(x) < 0$ when $x < 1.381966011$ and when $x$ in $[2.500000000, 3.618033989]$, $0 < D(f)(x)$ when $x$ in $[1.381966011, 2.500000000]$ and when $3.618033989 < x$, we know that the increasing intervals are $[1.381966011, 2.500000000]$ and $[3.618033989, \infty)$, the decreasing intervals are $[-\infty, 1.381966011]$ and $[2.500000000, 3.618033989]$.

Combining all this information, you can sketch the graph of this function. More details for sketching graphs for functions will come in later labs.

2. Rate of Change

The physical meaning of the first derivative is the rate of change. For a function defined by $y = f(x)$, $D(f)(a)$ is the instantaneous rate of change of the value $y$ with respect to the variable $x$ at $x = a$. If the value $y$ represents position and the variable $x$ represents time, then $D(f)(a)$ is the velocity at $x = a$. If $y$ represents the size of a population and $x$ represents time, then $D(f)(x)$ is the population growth rate. With these interpretations, we can solve many real problems related to rates.

**Example 1.** Find the instantaneous population rate in persons per minute at time $t = 10.1$ minutes if the population growth is approximated by the function $P = 3e^t + 1000$.

```plaintext
> P:=t->3*exp(t)+1000;
> D(P)(10.1);
```

**Example 2.** A spherical snowball is melting in such a way that its radius is decreasing at a rate of $\frac{1}{10}$ cm/min. At what rate (change per unit of time) is the volume changing when the radius is 5 cm?

First we need to get the volume function (related to the radius).

```plaintext
> V:=r->4/3*Pi*r^3;
```

Here, $V$ is a function of the radius and not of time. We cannot find the direct relationship between the volume and the time. But if we can compute the radius $r$ as a function of time, say $r = r(t)$, then the volume at time $t$ is $V = V(r(t))$ -- a composite function.

```plaintext
> with(student): V2:=makeproc(V(r(t)),t);
> D(V2)(t);
> compute the rate of volume change with respect to time t
```

For this question, we know that $r(t) = 5$ and $D(r)(t) = -\frac{1}{10}$, which is enough to compute the required rate.

```plaintext
> subs({r(t)=5,D(r)(t)=-1/10},%);
> evalf(%);
```
Thus the answer is: when the radius is 5 cm, the rate of change of the volume is 
\(-31.41592654 \text{ cm}^3/\text{min}\).
The main difficulty in a related rates problem usually lies in translating the words of 
the problem into mathematical equations between variables. Once the equations are 
written down, Maple can be helpful in manipulating them.

**Example 3.** The volume of a spherical balloon is increasing at a rate of 10 \( \text{ cm}^3/\text{s} \). 
How fast is the radius increasing when the surface area is 200 \( \text{ cm}^2 \)?

We are interested in the following quantities: volume \( V \), area \( A \), radius \( r \) of the 
balloon, and time. If we regard the radius as a function of time \( t \), then the functions 
\( V(t) \) and \( A(t) \) can be expressed as:

\[ V := r \rightarrow \frac{4}{3} \pi r^3; \]
\[ V(r(t)); \]
\[ V2 := \text{makeproc}(V(r(t)), t); \]
\[ D(V2)(t); \]
\[ A := r \rightarrow 4 \pi r^2; \]
\[ A2 := \text{makeproc}(A(r(t)), t); \]

From the assumption, we know the rate of change of the volume and the value of the 
radius, i.e. \( D(V)(t) \) and \( A(t) \) are known, we need to work out the rate of change of 
the radius \( D(r)(t) \). We have the relationship: 1) \( D(V)(t) = 4 \pi r(t)^2 D(r)(t) \) and 2) 
\( A = 4 \pi r^2 \). The second expression can be solved to find the radius since \( A = 200 \).

Now consider the first expression, \( D(V)(t) \) and \( r(t) \) are known, thus \( D(r)(t) \) can be 
found.

\[ r1 := \text{solve}(A(r) = 200, r); \]
\[ r1 := r1[1]; \]
\[ \text{only the positive one has physical meaning} \]
\[ \text{rprime:=solve}(D(V2)(t)=10,D(r)(t)); \]
\[ \text{subs}(r(t)=r1,rprime); \]

Hence, the answer to this question is: the radius is increasing at \( \frac{1}{20} \text{ cm/s} \) when the 
surface area is 200 \( \text{ cm}^2 \).

3. Exercises

- 1. Define the function \( f \) by \( f(x) = (x - 2)(x - 3)(x - 4)(x - 5) \). Use plots 
to investigate the relationship between \( f \) and its derivative. Use \( D(f)(x) \) to 
identify the regions where 1) \( f \) is increasing; 2) \( f \) is decreasing; 3) \( f \) has a 
local max or local min.

- 2. Use plots to find where the function \( f(x) = -(x - 3)^2 + 7x - 4 \) is maximized. 
Use \( D(f)(x) \) to locate this maximum precisely.
3. Assume the position of a particle is given by the function \( s(t) = t^3 - 7t^2 + 8t \), where \( t \) is measured in seconds and \( s \) is measured in meters. Plot the position of the object as a function of time for the first six seconds. Find the velocity of the particle at \( t = 3 \) sec. How does this differ from the speed of the object at \( t = 3 \)? At what time does the object halt to return to the origin for the second time?

4. Assume that the population of a certain country is growing at a rate of 2.7% per year. The population of that country in 1980 was 23 million, so that the population in millions after \( t \) years is given by \( p(t) = 231.027^t \). Plot the graph of this population function over 30 years. What population does this model predict for the year 2010? Compare the instantaneous rate of population growth (increase/total population) at time \( t = 0 \) and at time \( t = 30 \).

5. If \( f, p \) and \( q \) are related by the lens equation \( \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \), find the rate of change of \( p \) with respect to \( q \)?

6. During a hail storm, small droplets of water freeze into hail stones and gradually grow in size as additional moisture condenses on their surface. If the radius of a hail stone is increasing at a rate of \( \frac{1}{10} \) cm/min, at what rate (change per unit of time) is the volume changing when the radius is 1 cm?

7. A conical paper cup is 10 cm high and its radius at the top is 4 cm. Water is poured into the cup at the rate of 2 cm\(^3\)/s. What is the rate of change of the area of the (top) surface of the water after 5 seconds? Assume that the cup's axis is vertical and it is initially empty?

8. A triangle has one vertex at (0,0). The other two have the same \( y \) coordinate and are on the ellipse \( 4x^2 + 3y^2 = 4 \). What is the maximum area of such a triangle?

**Lab 21. More on Applications of Derivatives**

In the previous Lab, we used information from the first derivative to sketch the graph of the original function successfully. However, for a more general function, or to get
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a more precise picture of the function, the above method is not enough. This lab
introduces the general strategy of sketching the graph of a function.

1. Theory of Sketching Graphs

The main idea is to use information from the original function, its first derivative
and its second derivative to discover the main features of the graph of a function.
These features include symmetry, intercepts, critical points, singular points, inflection
points, asymptotes, intervals of increase and decrease, and intervals of concavity up
and concavity down.

The function itself provides information on zero points \( \{ x \mid f(x) = 0 \} \) (the x-intercepts),
the value of \( f(x) \) at \( x = 0 \) (the y-intercept), symmetry ( \( f(x) \) is odd or even), singular
points \( \{ x \mid f(x) \text{ is undefined} \} \), and asymptotes (vertical asymptotes, horizontal asymptotes, slant asymptotes). From its
first derivative, we can find the intervals of increase and decrease, the critical points
\( \{ x \in \text{Dom}(f) \mid D(f)(x) = 0 \text{ or } D(f)(x) \text{ undefined} \} \), and the local maximum and local
minimum. This has been fully discussed in the previous Lab. The second derivative
provides information on intercepts, inflection points, intervals of concavity up and
down, and also on the local extremum.

We know that \( D(\sin)(\pi/2) = \cos(\pi/2) = 0 \) and \( (D^{(2)})(\sin)(\pi/2) = -\sin(\pi/2) = -1 \). Thus,
\( \sin(x) \) has a local maximum at \( x = \pi/2 \). Since \( D(\cos)(\pi) = -\sin(\pi) = 0 \) and \( (D^{(2)})(\cos)(\pi)
= -\cos(\pi) = 1 \), \( \cos(x) \) has a local minimum at \( x = \pi \).

\[ \text{steps} \]
\[ D(\sin)(\pi/2), (D@@2)(\sin)(\pi/2); \]
\[ D(\cos)(\pi), (D@@2)(\cos)(\pi); \]
\[ f(x) = x^3 \text{ doesn’t have any local extremum at } x = 0 \text{ since } D(f)(0) = 0 \text{ and } (D^{(2)})(f)(0) = 0. \text{ In fact, } x = 0 \text{ is an inflection point, because } (D^{(2)})(f)(x) < 0 \text{ when } x < 0 \text{ and } 0 < (D^{(2)})(f)(x) \text{ when } 0 < x. \]
\[ > f:=x->x^3; \]
\[ > D(f)(0), (D@@2)(f)(0); \]
\[ > (D@@2)(f)(x); \]

The domain of \( x^3 \) is \( \mathbb{R} \), it is an odd function since \( f(-x) = -f(x) \), and has no
singular points. \( \lim_{x \to -\infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = -\infty, \text{ and } f(0) = 0. \) With all
these features, it shouldn’t be hard to sketch the graph of this function. Your sketch
should be almost the same as the graph plotted by Maple.

\[ > \text{plot}(x^3,x=-\infty..\infty,y=-\infty..\infty); \]

2. Discover All the Features and Sketch the Graph

The general strategy for sketching the graph of a differentiable function is as follows:

Step 1. Determine the domain \( \text{Dom}(f) \) of the function, and find all the singular
points.

Step 2. Check the symmetry, is \( f(x) \) odd, even, or neither?
Step 3. Find all the zeroes, i.e. solve the equation \( f(x) = 0 \) for \( x \), and compute the value \( f(0) \). Thus we know where the graph of \( f(x) \) intersects the \( x \)-coordinate and the \( y \)-coordinate.

Step 4. Find all the asymptotes. There are three types of asymptotes you should look for: vertical, horizontal and slant asymptotes.

Step 5. Compute the first derivative. Solve the equation \( D(f)(x) = 0 \) to find the critical points. Remember to include the points where \( D(f)(x) \) is undefined.

Step 6. Use the first derivative test to find the intervals of increase and decrease.

Step 7. Find all the local extrema, either from the information from step 6 or from the second derivative test.

Step 8. Solve \( (D^2)(f)(x) = 0 \) to find all possible inflection points.

Step 9. Check the sign of \( (D^2)(f)(x) \) to find the intervals of concavity up and down, thus confirming the inflection points.

Step 10. Make a table of all the features from the previous 9 steps. Check if there are any contradictions.

Step 11. Use the information from the table to sketch the graph of the function. You can plot the function in Maple to verify your sketch.

Consider the function \( f(x) = \frac{x^4+3x^2-4}{x^3+2x^2-4} \). This is a rational polynomial. The domain of this function includes all real numbers except those such that \( x^3 + 2x^2 - 4 = 0 \).

Thus, \( \text{Dom}(f) = (-\infty, 1.130395435) \cup (1.130395435, \infty) \). Also, it says that there is only one singular point at \( x = 1.130395435 \). There is no obvious symmetry about this function (neither odd nor even). Next, we will find the intercepts.

Neglect the complex roots. The \( x \) intercepts are \((1,0)\) and \((-1,0)\). It is clear that \( f(0) = 1 \), so the \( y \) intercept is \((0,1)\).

There is a vertical asymptote at the singular point \( x = 1.130395435 \). Notice the difference between the ten-decimal representation \( sp \) and the exact value \( s_1 \). To compute the limit, we need to use the exact value of the singular point \( s_1 \). Now, calculate some limits to find the other two types of asymptotes.

No horizontal asymptote.

There is a possible slant asymptote.
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> limit(f(x)-a*x,x=-infinity); limit(f(x)-a*x,x=infinity);
> b:=%;
The slant asymptote is \( y = a \cdot x + b \), i.e. \( y = x - 2 \).

> normal(D(f)(x));
The first derivative

\[
D(f)(x) = \frac{x(x^5 + 4x^4 - 16x^2 - 3x^3 - 8 + 12x)}{(x^3 + 2x^2 - 4)^2},
\]

which has the same domain as the original function \( f(x) \) does. Thus the only possible critical points are the roots of \( D(f)(x) = 0 \).

> cp:=solve(D(f)(x)=0,x);
The command \texttt{solve} can not give you the general expression for all the roots. The above answer from Maple means that 0 is a root of the equation \( D(f)(x) = 0 \), and the other five roots are the roots of the polynomial equation

\[
x^5 + 4x^4 - 16x^2 - 3x^3 - 8 + 12x = 0.
\]

To find all the critical points, we need to go further to find these roots.

> eq:=x^5+4*x^4-16*x^2-3*x^3-8+12*x=0;
> fsolve(eq,x,complex);
> cp:=fsolve(eq,x);
gives all the real roots of a polynomial equation

There are two critical points \( x = 0 \) and \( x = 1.718148949 \). These two critical points, together with the singular point, divide the real line into four intervals. Check the sign of \( D(f)(x) \) in each interval. Pick one value of \( x \) from each interval to check the sign of \( D(f)(x) \) in each interval.

> D(f)(-1), D(f)(1), D(f)(1.5), D(f)(2);
Thus, the intervals of increase are \((-\infty, 0)\) and \((1.718148949, \infty)\), the intervals of decrease are \((0, 1.130395435)\) and \((1.130395435, 1.718148949)\). From this, we also know that \( x = 0 \) is a local maximum and \( x = 1.718148949 \) is a local minimum. Now to find the inflection points.

> solve((D@@2)(f)(x)=0,x);
Again, we need to use \texttt{fsolve} to find all the real roots. To avoid typing the polynomial again, we use the command \texttt{subs}.

> subs(RootOf=fsolve,%%);
> infl:=evalf(%%);
Two possible inflection points and the singular point, divide the real line into four intervals. Check the sign of \( (D^{(2)})(f)(x) \) on each interval.

> (D@@2)(f)(-4), (D@@2)(f)(-2), (D@@2)(f)(0), (D@@2)(f)(2);
We see that \( x = -3.466559909 \) and \( x = 1.589279030 \) are inflection points. The function is concave down on the intervals: \((-\infty, -3.466559909)\) and \((-1.589279030, 1.130395435)\), concave up on the intervals: \((-3.466559909, -1.589279030)\) and \((1.130395435, \infty)\).
Now make a table including all the intercepts, critical, singular and inflection points, the vertical asymptote, the slant asymptote, intervals of increase and decrease, the intervals of concave up and down. Finally, you are ready to sketch the graph of this rational function.

We will use Maple to plot this function to verify your sketch. To include all the main features, we need the $x$ interval at least in $[-3.466559909, 1.718148949]$, with some room to spare on both sides. We might try $[-5, 3]$. Since there is a vertical asymptote, we must choose the $y$ interval as well. We want to include at least the inflection points and the critical points: $f(-3.466559909) = -8.160502595$, $f(-1.589279030) = -3.360945208$, $f(0) = 1$, and $f(1.718148949) = 1.945299758$. So $y$ in $[-9,3]$ might seem reasonable. To get a better view of the whole graph, we will use $[-9,5]$ the $y$ interval. Since $\text{Dom}(f)$ of this function includes two separate intervals and there is one singular point, we need the $\text{discont=true}$ option to plot the graph. We will show the asymptotes as dotted lines, place diamonds at the critical points and circles at the inflection points. This can be done by the following commands.

```maple
> p1 := plot(f(x), x=-5..3,-9..5, discont=true):
> p2 := plot(x-2, x=-5..3, linestyle=3):
> p3 := plot([[sp,-9], [sp,5]], linestyle=3):
> p4 := plot([[0,f(0)],[cp,f(cp)]], style=POINT, symbol=DIAMOND):
> p5 := plot([[infl[1],f(infl[1])],[infl[2], f(infl[2])]], style=POINT, symbol=CIRCLE):
> plots[display]({p1,p2,p3,p4,p5});
```

3. Exercises

- 1. Follows the steps in section 2 to sketch the graphs of the following functions.
  a) $f(x) = \sqrt{(x-1)(x-2)(x-3)}$;
  b) $f(x) = (x+2)^{\frac{3}{2}} - (x-2)^{\frac{3}{2}}$;
  c) $f(x) = \frac{(x+1)^3}{(x-1)^2}$;
  d) $f(x) = \ln(x + \sqrt{x^2 + 1})$;
  e) $f(x) = (1 + x)^{\frac{1}{2}}$.

- 2. Analyze the following functions using the 10 steps discussed in section 2. Plot the graphs of these functions showing all intercepts, critical, singular, and inflection points and asymptotes.
  a) $\ln(1 + x^2)$;
  b) $\frac{x^2 + 1}{x + 1}$;
c) $x \ln(x)$

d) $\frac{1+x^3}{1+x^2}$.

- 3. Plot a graph of the function $f(x) = \frac{x^3+6\sqrt{1+x^2}}{2x-\sqrt{1+x^2}}$, showing all intercepts, critical, singular and inflection points and asymptotes.

- 4. Plot a graph of $f(x) = (x^3-x^2-x+1)^{\frac{1}{3}}$ showing all intercepts, critical, singular and inflection points and asymptotes. Where is the function increasing? decreasing? concave up? concave down? Hint: Maple returns a complex number for $t^{\frac{1}{3}}$ when $t<0$, which is not what we want. You can use $f(x)$ as above when $0 \leq x^3-x^2-x+1$, and $-(x^3+x^2+x-1)^{\frac{1}{3}}$ when $x^3-x^2-x+1 < 0$, and combine graphs with display (or use piecewise or interval).

**Lab 22. Introduction to Indefinite Integral**

In this Lab we present an introduction to the indefinite integral, its definition, its relationship with the derivative, its properties and its computation in Maple. We will learn to use Maple to do some integration. Maple is very good at integration. If there is an antiderivative that can be expressed in terms of the ordinary functions of calculus, Maple is likely to find it. This lab uses Maple to integrate automatically. However, more important and more useful ideas for using Maple to integrate, called integration techniques, are included in the next four labs.

1. Definition of the Indefinite Integral

The indefinite integral of $f(x)$ is denoted by $\int f(x) \, dx$, which, in fact, is the antiderivative of $f(x)$. If we regard the differential $\text{diff}()$ as an operator, then the integral operator $\text{int}()$ can be considered as the inverse of the differential operator. So the two operators cancel each other out.

\[
\text{diff(int(f(x),x),x)}; \\
\text{int(diff(f(x),x),x)};
\]

Based on this, there is always a way to verify integration result.

\[
\text{p:=int(5*x*(5*x^2+1)^6,x)};
\]

Maple returns a cumbersome-looking long expression for above integration. Are you sure it is right? Use its inverse operator diff to verify the result.

\[
\text{diff(p,x)};
\]

This should give back the original integrand $5x(5x^2+1)$. However, the result from the above command looks much more complicated than the original integrand.

\[
\text{expand(5*x*(5*x^2+1)^6)};
\]
Now this agrees with the answer of the previous command. It verifies the result of integration, also tells us the method used to compute the integration. Maple first expanded the integrand and then integrated each term, which is why the answer looks complicated. Definitely, you don’t want to do it this way by hand. There is a better way for this kind of integration, which will be explained in the next Lab.

All the basic integration rules can be translated into corresponding rules of differentiation. There are three elementary rules we all should keep in mind. The linearity rule corresponds to the linearity rule of differentiation, the substitution rule, which corresponds to the chain rule of differentiation, and the integration by parts rule, which corresponds to the product rule of differentiation, will be fully discussed in the next two Labs.

```
> expand(int(alpha*f(x)+beta*g(x),x));
> combine(%);
```

2. Integration in Maple.
As we have seen, the `int` command is what is normally used in Maple to find an antiderivative, the indefinite integral.

```
> int(0,x);
> int(1,x);
> int(x^mu,x)
> ;
```

Maple won’t give you the general answer for the indefinite integral as $\int f(x) \, dx = f(x) + C$. Instead, it only works out one antiderivative $f(x)$. To complete the answer, you must add to it the constant number $C$. For example, $\int 0 \, dx = C$, $\int 1 \, dx = x + C$ and $\int x^\mu \, dx = \frac{x^{\mu+1}}{\mu+1} + C$.

There are some formulas for the differentiation of elementary functions. Once you remember these formulas, computing the derivative of any function becomes easier. Also, it will not take you too much time to do the integration by hand.

Integration is closely related to differentiation. It also has these important and useful formulas, which are listed on page 180 of the textbook. Use Maple to help you to remember these formulas.

```
> int(1/x,x);
> int(1/(1+x^-2),x);
> int(a^-x,x);
> int(exp(x),x);
> int(sin(x),x);
> int(cos(x),x);
> int(1/sin(x)^-2,x);
> int(1/cos(x)^-2,x);
> int(sinh(x),x);
> int(i/cosh(x),x);
```
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\[
> \text{int}(1/\sinh(x)^2, x);
> \text{int}(1/\cosh(x)^2, x);
\]
Later on, you will learn to integrate directly (with Maple acting as your assistant and carrying out the detailed manipulations) by using the techniques you learned in your course. Sometime you do not want Maple to try to do the integration itself, you need to use \text{Int} instead.
\[
> \text{Int}(x\sin(x), x);
\]
“\text{Int}” is an “inert” function, producing an expression that can be manipulated in various ways. Maple has many examples of this kind of “inert” functions, such as “\text{Sum}”, “\text{Limit}”, “\text{Diff}”, etc.
\[
> \text{Limit}(\sin(x)/x, 0);
> \text{Diff}(\sin(x), x);
> \text{Sum}(1/k, k=0..n);
\]
3. Exercises
Use Maple to compute the following indefinite integrals. Then verify the answers by differentiating the resulting expression, when necessary apply the basic algebraic and trigonometric transformations.

- 1. \[\int \frac{1}{\sin(x)\cos(x)^2} \, dx\]
- 2. \[\int (x^2 + 5)^3 \, dx\]
- 3. \[\int \tan(x)^2 \, dx\]
  Hint: Ask Maple to show every detail of its computation.

Lab 23. Integration by Substitution
This Lab and the next three labs will concentrate on some classical integration techniques with Maple doing the algebraic manipulations. The basic framework for these Labs is as follows: start with an assigned integral expression, apply \text{Int}, and manipulate it in various ways. When each integrand in our expression is one of the standard integrals discussed in the previous Lab (Lab 22) that every calculus student should have memorized, or a constant of one, we will allow \text{value} to perform the integration. Then we will perform back-substitution and simplification as necessary to produce the final answer.

1. Substitution Technique
The substitution technique is also called the change of variable technique, which can be done by using the `changevar` command. However some commands are from the `student` package, which have first to be loaded in order to do anything else in this Lab.

```plaintext
> with(student);
```

Suppose now we have an expression \( \int f(x) \, dx \). To change the variable of integration from \( x \) to \( u \), which are related by an equation \( g(x) = h(u) \), we will use

```plaintext
> changevar(g(x)=h(u), Inte, u);
```

This includes both the "direct" substitution \( g(x) = u \) and the "inverse" substitution \( x = h(u) \). The third input is the name of the new variable, which should not be present in the expression. In Maple, the substitution technique is used as follows. Suppose that we want to use the substitution technique to evaluate \( \int 2x \sqrt{1 + x^2} \, dx \).

First, use \textbf{Int} to produce an integral expression.

```plaintext
> Inte1:=Int(2*x*sqrt(1+x^2),x);
```

Then, select a substitution: \( 1 + x^2 = u \). We will have \( 2x \, dx = du \). It is not hard to change the original integral to the new one \( \int \sqrt{u} \, du \) which is easy to compute. You can do this transformation by hand. But Maple is very good at this algebraic manipulation. Why not ask Maple to do it?

```plaintext
> Inte2:=changevar(1+x^2=u,Inte1, u);
```

The substitution \( 1 + x^2 = u \) reduces the integration problem. The new integration problem is essentially equivalent to the original one. However, we have simplified our task to that of finding an antiderivative for \( \sqrt{u} \). This is now an "elementary integral", so we use \textbf{value}.

```plaintext
> value(%);
```

To recover the value of the original integral, we substitute back to get this in terms of \( x \), using the relation \( u = 1 + x^2 \). We can use \textbf{subs} here.

```plaintext
> subs(u=1+x^2,%);
```

This gives us the final answer for the original integral: \( \int 2x \sqrt{1 + x^2} \, dx = \frac{2(1+x^2)^{3/2}}{3} \).

There is an independent way of verifying that this is a correct solution. All we need to do is to differentiate the final solution with respect to \( x \).

```plaintext
> diff(%o,x);
```

We should get something that can be simplified to the original integrand. This method of verifying the solution for integration can be used at any time.

Now, we summarize the substitution technique. It involves five steps:

1. Use \textbf{Int} to produce an integral expression.
2. Select a suitable substitution \( u = g(x) \).
3. Rewrite the integral in terms of \( u \), using \textbf{changevar}.
4. Solve the new integral.
5. Substitute back to get the solution in terms of \( x \) by using \( u = g(x) \).
All these can be done by hand or by Maple. No matter which way you choose to manipulate the computation, the second step may be repeated several times before a final answer is obtained.

A wide variety of relations \( u = g(x) \) can be used to define the substitution, although only some choices will lead to a simplification of the problem. Thus, the second step requires practice and insight, while the other steps are completely mechanical for any given substitution.

When you do the third step by hands, please notice that the substitution is more than the replacement of \( g(x) \) by \( u \). The expression \( dx \) must also be written in terms of \( du \), and if necessary, the equation \( x = g^{-1}(x) \) must be used to move any further references to \( x \). However, all these will be automatically handled when the Maple command \texttt{changevar} is used.

2. More Examples

Example 1. Evaluate \( \int \frac{x}{1+x^4} \, dx \).

\begin{verbatim}
> e1:= Int(x/(1+x^4),x);

Choose a substitution: \( x^2 = t \), then \( 2x \, dx = dt \). Manipulate this substitution by Maple.

> e2:=changevar(x^2=t,e1,t);

This is now an elementary integral, which should have been memorized by you. Now we use \texttt{value} to get the result.

> value(e2);

Finally, we substitute back to get the answer for the original integral.

> subs(t=x^2,%);

You may use the integral operator’s inverse operator, the differential operator, to verify the above solution.

> diff(% ,x);

Now, when you copy down the solution for the original integral, don’t forget to add a constant \( C \) since here you are giving the answer for an indefinite integral. Hence, the final answer should be: \( \int \frac{x}{1+x^4} \, dx = \arctan\left(\frac{x^2}{2}\right) + C \).

The primary goal of this substitution technique is to reduce the original integral to an elementary integral or a constant multiple of one. The success of this depends on our ability to recognize appropriate patterns. A good choice for \( u = g(x) \) will usually correspond to \( D(g)(x) \) occurring as a factor of the integrand and \( g(x) \) occurring as an argument of a function this is present.

Example 2. Find \( \int x^3 \cos(x^4 + 2) \, dx \).

First, change the integral to an expression using \texttt{Int}.

\begin{verbatim}
> e1:=Int(x^3*cos(x^4+2),x);

Now we try to find a suitable substitution. Since \( x^3 \) occurs as a factor of the integrand and \( x^4 \) occurs in the argument to cosine, you might want to try \( x^4 = u \).
The new integral appears simpler but is not yet the one that we have already solved. We need to do further substitution to simplify the argument of cosine. A good candidate for this is $u + 2 = v$.

Now this gives the standard integral $\int \cos(v) \, dv$ which is in fact the sine function.

We have subsituted twice, so we recover the answer to the original problem by using both of the substitutions: $v = u + 2$ and $u = x^4$. With Maple, we can finish this in one step.

Hence the answer of the original problem is: $\int x^3 \cos(x^4 + 2) \, dx = \frac{\sin(x^4+2)}{4} + C$.

In fact, there is one substitution that can do all the work.

This is a better choice of substitution.

3. Exercises

Compute the following integrals using the substitution technique.

• 1. $\int 5 \, x (5 \, x^2 + 1)^6 \, dx$.

• 2. $\int \sqrt{3 - \frac{t}{2}} \, dt$.

• 3. $\int \frac{x+3}{(x^2+6 \, x)^2} \, dx$.

• 4. $\int x^5 \, \sqrt{x^3 + 1} \, dx$.

• 5. $\int \frac{2 \, x+1}{\sqrt{6 \, x^2+6 \, x+1}} \, dx$.

• 6. $\int \frac{x^2}{\sqrt{1+x^6}} \, dx$.

• 7. $\int \frac{1}{3 \, \sqrt{1+x^2}} \, dx$. 
8. \( \int 6x^5 \sin(x^6 - 3) \, dx \).

9. \( \int \frac{\sqrt{2}(3x^3 + 2x)}{x} \, dx \).

10. \( \int x^2 \sin(x^3) \, dx \).

11. \( \int x^2 \sqrt{x^3 + 1} \, dx \).

12. \( \int (3x^2 + 2x) \sqrt{x^3 + x^2} \, dx \).

13. \( \int (10x + 1)(5x^2 + x + 3)^3 \, dx \).

14. \( \int 3(x^3 + 1)x^2 \, dx \).

**Lab 24. Integration by Parts**

The second integration technique is integration by parts. It is most used when we are unable to find a helpful change of variables. Like integration by substitution, integration by parts is a tool that can be used to examine an integration problem from different points of view. Some points of view are more helpful than others. As with any tool, a lot of practise is necessary in order to use it effectively.

1. **Technique of Integration by Parts**

Integration by parts arises from a careful examination of the standard rule for computing derivatives of products. The theory of this has already been fully discussed in the textbook. What we need to do here is to learn how to manipulate this using Maple.

In Maple, integration by parts is available as part of the *student* package. It accepts two arguments. The first is an expression containing an unevaluated integral and the second indicates the part that has to be regarded as an antiderivative. Suppose *expr* is an expression containing an *Int* which can be written as \( \int f(x)D(g)(x) \, dx \). Then to produce \( f(x)g(x) - \int D(f)(x)g(x) \, dx \), we use

```maple
> expr := Int(f(x)*D(g)(x),x);
> with(student):
```
If we assume $U = f(x)$ and $dV = D(g(x))\, dx$, then the above formula says $\int U \, dV = UV - \int V \, dU$. Before you try the command `intparts` on any specific problem, you should be sure that the antiderivative of $dV$ (i.e. $g(x)$) can be found.

The trick to successfully using this technique is to know what to choose for $U$. Once we have chosen $U$, $dV$ is also determined, which in turn determines $V$. For example, consider $\int x \sin(x) \, dx$. There are two obvious choices for $U$, $U = \sin(x)$ (so $dV = x \, dx$) or $U = x$ (so $dV = \sin(x) \, dx$). Let us try both of them and see which one is better.

```maple
> I1 := Int(x*sin(x), x);
> I2 := intparts(I1, sin(x));
> I3 := intparts(I1, x);
```
Which answer is simpler? which of the new integrals can be computed by hand? The second is clearly more helpful since we know that the antiderivative of $-\cos(x)$ is $\sin(x)$.

```maple
> value(%);
```
Hence the result is: $\int x \sin(x) \, dx = -x \cos(x) + \sin(x) + C$.

Let us take a closer look at the above example and do some analysis. The derivative of the sine function involves the cosine function, which does not appear in the integrand, so a straightforward change of variables is unlikely to be effective in this example. We know how to integrate and differentiate both factors $x$ and $\sin(x)$ of the integrand. But if we take $U = \sin(x)$, then $V = \frac{x^2}{2}$ and $dU = \cos(x)$, so after integration by parts, we will obtain more complicated integral $\int \frac{x^2 \cos(x)}{2} \, dx$. On the other hand, if we take $U = x$, then $V = -\cos(x)$ and $dU = 1$, thus after integration by parts we get the integral $\int -\cos(x) \, dx$ which is one of the standard integrals. Therefore, for integrals of the form $p(x) \sin(x)$ or $p(x) \cos(x)$ where $p(x)$ is polynomial in $x$, we can reduce the degree of the polynomial by regarding $p(x)$ as an antiderivative.

Sometimes, it is helpful to regard a single term as a product multiplied by 1. In this way, we can still try integration by parts on an integral of a single term. For example, consider the integral $\int \ln(x) \, dx$. We regard the integrand as $1 \cdot \ln(x)$. Take $U = \ln(x)$, so $dV = 1$, thus $V = x$ and $dU = \frac{1}{x}$. Can you try this in the other way taking $U = 1$? What will happen?

```maple
> I1 := Int(ln(x), x);
> I2 := intparts(I1, ln(x));
> value(%);
```
The answer is now: $\int \ln(x) \, dx = \ln(x) \, x - x$.

### 2. More Examples

As we have said before, we need to practice, we need to try this technique on many problems. Also, memorizing some typical methods for some general types of integral
problems will be very helpful. For instance, from the example $\int x \sin(x) \, dx$, we can see that for integrals of type $\int p(x) \sin(x) \, dx$ or $\int p(x) \cos(x) \, dx$, we should always try $U = p(x)$. For $\int \arctan(x) \, dx$ or any single term’s integral, we should try the method that works for $\int \ln(x) \, dx$. If we work out the integral $\int x^3 \ln(x) \, dx$, we should be aware that the method probably works for integrals of type $\int p(x) \ln(x) \, dx$ where $p(x)$ is a polynomial in $x$.

For example, let us evaluate $\int (x^2 + 1)^2 \ln(x) \, dx$.

```maple
> e1:=Int((x^2+1)^2*ln(x),x);
> e2:=intparts(e1,ln(x));
> value(%);
```

So the answer is $\int (x^2 + 1)^2 \ln(x) \, dx = \ln(x) \left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right) - \frac{x^5}{25} - \frac{2x^3}{9} - x + C$.

For some integrals, applying integration by parts only once is not enough. A repeating application of the integration by parts rule is called the generalized integration by parts rule. For example, evaluate the integral $\int x^3 \sin(x) \, dx$.

```maple
> e1:=Int(x^3*sin(x),x);
> e2:=intparts(e1,x^3);
> e3:=intparts(e2,x^2);
> e4:=intparts(e3,x);
> value(%);
```

For integrals of type $\int x^k \ln(x)^m \, dx$, take $U = \ln(x)^m$, then $U = \ln(x)^{m-1} \ldots U = \ln(x)$. For type $\int x^k \sin(b \, x) \, dx$, $\int x^k \cos(b \, x) \, dx$ or $\int x^k e^{(a \, x)} \, dx$, take $U = x^k$, then $U = x^{k-1} \ldots U = x$. All these types need the generalized integration by parts rule. Also, for the more types $\int p(x) e^{(a \, x)} \, dx$, $\int p(x) \sin(b \, x) \, dx$ or $\int p(x) \cos(b \, x) \, dx$, we have to apply the generalized integration by parts rule by first taking $U = p(x)$, and then $U = p'(x) \ldots U = p^n(x)$ where $n$ is the degree of $p(x)$.

Evaluate the integral $\int (x^3 + x^6 + 1) \sin(x) \, dx$.

```maple
> e1:=Int((x^3+x^2+1)*sin(x),x);
> e2:=intparts(e1,x^3+x^2+1);
> e3:=intparts(e2,3*x^2+2*x);
> e4:=intparts(e3,x+2);
> value(%);
```

Finally, let us take a look at an interesting example: $\int e^x \sin(x) \, dx$. If we repeatedly take the derivative of $\sin(x)$, we eventually get back to $\sin(x)$. Furthermore, $e^x$ is unchanged by integration or differentiation with respect to $x$. This means that after the generalized integration by parts, the original integral will appear as part of the answer.

```maple
> e1:=Int(exp(x)*sin(x),x);
> e2:=intparts(e1,exp(x));
> e3:=intparts(e2,exp(x));
> The original integral is then part of the expression.
```
> isolate(e1=expand(e3),e1);

Also, you can try:

> e2:=intparts(e1,sin(x));
> e3:=intparts(e2,cos(x));
> isolate(e1=expand(e3),e1);

3. Exercises

• 1. For each of the following integrals, make two different choices of $U$. For each choice, apply integration by parts once. Which choice of $U$ is better in each case, or do both choices work well? Use your preferred choice of $U$ to complete the computation of the integral. You may have to apply the generalized integration by parts rule, and if so, show each step, including each choice of $U$.

   a). $\int x^2 \cos(x) \, dx$.
   b). $\int x^2 e^{(2x)} \, dx$.
   c). $\int e^x \sin(x) \, dx$.

• 2. Use integration by parts with $U = \ln(x)$ to derive a formula for $\int x^n \ln(x) \, dx$. Check that the formula fails when $n = -1$. Can you explain what step in your derivation fails when $n = -1$? Is there any other value of $n$ for which the formula also fails? What is the correct answer for $\int x^{(-1)} \ln(x) \, dx$?

• 3. Transform the integral $\int u \cos(u) \, du$ into an integral that does not involve a product using integration by parts. Verify that the value of the resulting integral is a valid antiderivative for $u \cos(u)$.

• 4. Integration by parts can be used to re-express the integral $\int x^2 \ln(x) \, dx$ in many different forms. Try taking $U = x$, $U = x^2$, $U = x^3$, $U = \ln(x)$ and $U = x \ln(x)$. In each case, identify the integrals and derivatives that need to be computed to carry out the transformation. Can you identify a pattern?

• 5. Complete the construction of an antiderivative of $x^2 \ln(x)$ for each of the transformations attempted in the previous question. This will involve evaluating directly any new integrals that are produced and isolating all the terms involving the original integral on one side of the equation.
• 6. Use integration by parts and a change of variables to evaluate
\[ \int x^2 \arctan(x) \, dx \] in terms of a logarithm term.

• 7. Evaluate the following integrals and show each step of your computation. Only use \texttt{int} or \texttt{value} for elementary integrals that you have memorized.
  a). \[ \int \arctan(\sqrt{x}) \, dx. \]
  b). \[ \int x^2 e^x \, dx. \]
  c). \[ \int x^2 \arctan(x) \, dx. \]
  d). \[ \int x^4 \ln(x)^2 \, dx. \]
  e). \[ \int x^6 \sin(2x) \, dx. \]
  f). \[ \int x^7 \cos(3x) \, dx. \]
  g). \[ \int (x^2 + 1)^3 e^{(a x)} \, dx. \]
  h). \[ \int (5x^2 + 4x + 1)^2 \sin(2x) \, dx. \]
  i). \[ \int (x - 1)^2 \cos(4x) \, dx. \]
  j). \[ \int e^x \cos(bx) \, dx. \]
  k). \[ \int e^x \sin(bx) \, dx. \]
  l). \[ \int \frac{x^2 + 3}{\sqrt{(2x-5)^3}} \, dx. \]
  m). \[ \int \frac{1}{a^2 \sin(x)^2 + b^2 \cos(x)^2} \, dx. \]
  n). \[ \int x^\left(\frac{1}{7}\right) \ln(x)^2 \, dx. \]
  o). \[ \int 3^x \cos(x) \, dx. \]

\textbf{Lab 25. Integration of Rational Functions}

The third main integration technique is the partial fraction decomposition. This decomposition should be considered whenever a quotient of polynomials is involved.

1. Computing a Partial Fraction Decomposition

There are two ways to compute a partial fraction decomposition for a rational function. One way is to do it ourselves using the general form of the decomposition discussed in the textbook. The other way is to call the built in function from Maple.

\textbf{Example 1.} Find a partial fraction decomposition for

\[ f = \frac{x^5 - 9x^4 + 26x^3 - 18x^2 - 24x + 23}{x^4 - 10x^3 + 35x^2 - 50x + 24}. \]
\[ f := x \rightarrow \frac{x^5 - 9x^4 + 26x^3 - 18x^2 - 24x + 23}{x^4 - 10x^3 + 35x^2 - 50x + 24}; \]

Please notice that the degree of the numerator is 5 which is greater than the degree of denominator 4. This function is not a proper rational function.

\[ f_1 := \text{quo}(\text{numer}(f(x)), \text{denom}(f(x)), x); \]

This command accepts two polynomials as arguments and divides the second polynomial into the first. It reports the result but does not report the remainder. To catch the remainder term, we need to use:

\[ \text{remf} := \text{rem}(\text{numer}(f(x)), \text{denom}(f(x)), x); \]
\[ f_2 := \text{remf}/\text{denom}(f(x)); \]
\[ \text{factor}(\text{denom}(f_2)); \]

\( f_1 \) contains the polynomial part of \( f(x) \) and \( f_2 \) is a proper rational function now.

From the textbook, we know that the general form of the decomposition for \( f_2 \) is

\[
-\frac{1}{x-1} + \frac{3}{(x-2)(x-3)(x-4)} = \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{x-3} + \frac{d}{x-4}. \]

We need to find out the values of \( a, b, c \) and \( d \).

\[ \text{eq} := f_2 = a/(x-1) + b/(x-2) + c/(x-3) + d/(x-4); \]
\[ \text{eq} \ast \text{denom(lhs(\%))}: \text{eq1} := \text{solve}(\%); \]

The result is a polynomial equation in \( x \) that must be true for all values of \( x \). Such an equation is known as a polynomial identity in \( x \), and it can be used to find the values of \( a, b, c \) and \( d \). By subtracting the right-hand side from both sides of the equation, the right-hand polynomial becomes zero. Thus, all the coefficients on the left-hand side of the new equation must be zero.

\[ (\text{lhs} - \text{rhs})(\text{eq1}); \]

Collect according to the powers of \( x \) by the command \text{collect}.

\[ \text{collect}(\%, x); \]

Get the set of coefficients of the powers of \( x \).

\[ \text{cf} := \{\text{coeffs}(\%, x)\}; \]

All these coefficients must be zero. Solve the equation \( cf = 0 \) for \( a, b, c \) and \( d \).

\[ \text{solve}(\text{cf}); \]

Finally we get the partial fraction decomposition for \( f(x) \).

\[ f(x) = x + 1 + \frac{1}{6(x-1)} + \frac{-\frac{1}{2}}{x-2} + \frac{-\frac{5}{2}}{x-3} + \frac{23}{(x-4)6}. \]

For each partial fraction decomposition we compute, we have a polynomial identity to solve. Rather than carry out the full procedure as just outlined in the above, we can proceed as follows.

\[ \text{identity(\text{eq1}, x)}; \]

It tells that \( eq \) is an identity for \( x \). Then the solution to the identity can be obtained by executing the command \text{solve}.

\[ \text{solve}(\%); \]

Finding the partial fraction decomposition of a function by the above procedures is a straightforward exercise, but it could be a very tedious one. We can make Maple to do
it for us by the command `convert`. The `convert` command with the option `parfrac` converts a rational function to partial fractions when the factors of the denominator have rational coefficients. The third input of the `convert` command tells it the name of the variable being used. Note that this is the second time we have used `convert`. Please use the online help to find more details.

Let us solve the question in Example 1 again. But this time we will use `convert`.

```maple
> F:=convert(f(x),parfrac,x);
```

It is simple. One command finds the partial fraction decomposition.

2. More Examples of Integration of Rational Functions

Example 2. Evaluate \( \int \frac{x^7+2x^6-21x^2-233x-1809}{x^6-2x^4+48x^3+x^2-48x+576} \, dx \).

```maple
> ra:=(x^7+2*x^6-21*x^2-233*x-1809)/(x^6-2*x^4+48*x^3+x^2-48*x+576);
> fra:=convert(ra,parfrac,x);
```

The first four terms of the decomposition are simple to integrate, but the next two terms require some work. We have to break `fra` into pieces. This can be done using the `op` command, which extracts one of the operands making up the expression. In this case, `fra` is a sum of six terms, each of which is an operand of `fra`, and these can be obtained by `op(1,fra), op(2,fra),... op(6,fra)`. We first take the sum of the first four operands of `fra` and then integrate.

```maple
> op(1,fra)+op(2,fra)+op(3,fra)+op(4,fra);
> inte1:=int(%,x);
```

Notice that in the answer, \( \ln(x+3) \) should be \( \ln(|x+3|) \). Maple wrote the antiderivative of \( \frac{1}{x+3} \) as \( \ln(x+3) \). This is because Maple works with complex numbers rather than just real numbers. But we need to change \( \ln(x+3) \) into \( \ln(|x+3|) \).

```maple
> inte1:=subs(ln(x+3)=ln(abs(x+3)),%);
```

Now, we work on the last two operands.

```maple
> inte2:=Int(op(5,fra)+op(4,fra),x);
> with(student): completesquare(inte2,x);
```

The `completesquare` command in the `student` package will complete the square in any quadratic in the variable that occurs in `expr`. The answer shows which change of variables to use.

```maple
> changevar(x-3/2=t,%,t);
> inte2:=expand(%%);
> : to deal with the terms seperately.
> inte2a:=value(op(1,inte2));
> :elementary integral.
```

We can use the substitution \( t^2 + \frac{23}{4} = s \) on the second and fourth terms of `inte2`.

```maple
> changevar(t^2+23/4=s,op(2,inte2)+ op(4,inte2),s);
> value(%%);
```
Again \( \ln(s) \) should be \( \ln(|s|) \). However, since \( s = t^2 + \frac{23}{4} \), we don’t need to make any change.

\[
\text{inte2b} := \text{subs}(s=t^2+23/4,\%)\;
\]
The rest is the third term. Try the substitution \( t = \sqrt{\frac{23}{4}} \tan(s) \).

\[
\text{changevar}(t=\sqrt{23/4}*\tan(s), \text{op}(3,\text{inte2}),s);\;
\]
\[
\text{simplify}(\%);\;
\]
\[
\text{subs}(1/(1+(\tan(s))^2)=(\cos(s))^2,\%);\;
\]
\[
\text{combine}(\%\text{,trig);}\;
\]
\[
\text{to write}\;
\]
\[
\cos(s)^2 = \frac{\cos(2s)+1}{2}.\;
\]
\[
\text{value}(\%);\;
\]
Now substitute back to write the answer in terms of \( t \).

\[
\text{inte2c} := \text{eval(subs}(s=\text{arctan(}\sqrt{4/23}*t\text{), }\%\text{));}\;
\]
Combine all the parts together, and change variables from \( t \) back to \( x \).

\[
\text{fe} := \text{inte1}+\text{subs}(t=x-3/2,\text{inte2a+inte2b+ expand(inte2c)});\;
\]
We still need some simplification for the 6th, 7th and 8th term. The final answer will be:

\[
\text{fe} := \text{sum}(\text{op}(i,\%),i=1..5)+\text{simplify(}\text{op}(6,\%))\text{+simplify(}\text{op}(7,\%)+\text{op}(8,\%));}\;
\]
In the next example, there is something different from our usual work.

**Example 3.** Evaluate \( \int \frac{x^2+1}{x^4-x^2+1} \, dx \).

\[
\text{ra} := (x^2+1)/(x^4-x^2+1);\;
\]
\[
\text{convert(ra,parfrac,x);}\;
\]
The factors exist, but not all involving rational numbers. We use \text{solve} to find what kind of coefficients they might have.

\[
\text{solve}(\text{denom(ra)=0,x});\;
\]
The roots involve \( \sqrt{3} \), which means the coefficients of the factors can involve \( \sqrt{3} \) as well as rational numbers.

\[
\text{convert(ra,parfrac,x,}\sqrt{3});\;
\]
The rest of the integration is routine: use \text{completesquare}, then change variables in each of the terms. The final answer is \( \int \frac{x+1}{x^2-x+1} \, dx = \arctan(2x - \sqrt{3}) + \arctan(2x + \sqrt{3}) + C \).

3. Exercises

- 1. Find \( \int \frac{1}{x+2\sqrt{x-3}} \, dx \).
2. Ask Maple to do the decomposition, then integrate the resulting expression by hand for the integral $\int \frac{1}{x^3-3x^2+4} \, dx$.

3. Find the partial fraction decomposition by the first method discussed in section 1 for the rational function $r = \int \frac{x^3+3x^2-x+1}{x^2+6x+8} \, dx$. Then integrate each term by hand. You can have Maple integrate $r$ directly to check your result.

4. For each of the following functions $f(x)$, use Maple to find the partial fraction decomposition. Using this decomposition, compute $\int f(x) \, dx$ by hand, showing your work.
   a). $f(x) = \frac{x+3}{2x^2-3x-2}$.
   b). $f(x) = \frac{x}{x^3-2x^2+x-2}$.
   c). $f(x) = \frac{x^4+1}{x^3+1}$.
   d). $f(x) = \frac{(\sin(x)-2)\cos(x)}{\sin(x)^2+4\sin(x)+3}$.
   e). $f(x) = \frac{\sin(x)}{\cos(x)^3+\cos(x)}$.

5. Reformulate the integral $\int \sqrt{16-x^2} \cdot x(-2) \, dx$ as an integral involving $\sqrt{1-u^2}$ for some suitable choice of $u$.

6. Describe how to use a change of variables to reexpress $\int \frac{x(-2)}{\sqrt{x^2+4}} \, dx$ as an integral of $\sec(v)^2$.

7. By completing the square and using a change of variables, reformulate $\int \frac{3x}{\sqrt{8-2x-x^2}} \, dx$ as an integral involving a denominator of the form $\sqrt{1-v^2}$.

8. Use the first method in section 1 to find the partial fraction decomposition for $\frac{x^3-5x^2+8x-2}{x^4-2x^3-13x^2+14x+24}$. Use the partial decomposition to evaluate the integral $\int \frac{x^3-5x^2+8x-2}{x^4-2x^3-13x^2+14x+24} \, dx$. 
Lab 26. Integration of Trigonometric and Hyperbolic Functions

Several methods of integration of trigonometric and hyperbolic functions are fully discussed in the textbook. In this Lab we show how to use Maple as an assistant in performing the algebraic computations. There are some examples for each type of integral. Finally, we will deal with the reduction formula.

1. Integration of Trigonometric Functions

The first type of integral is of the form \( \int \sin(x)^m \cos(x)^n \, dx \), where \( m \) and \( n \) are rational numbers. Let us first solve the easiest case when \( m \) and \( n \) are positive numbers. We will use trigonometric transformation to reformulate the integral.

Example 1. Find \( \int \sin(x)^5 \cos(x)^2 \, dx \).

First, we use the command `combine`, and choose the option `trig`, which tells Maple to use trigonometric transformation.

\[
> \text{Inte}:=\text{Int}(\sin(x)^5\cos(x)^2,x);
\]

\[
> \text{combine}(\%,\text{trig});
\]

The new integral almost looks like a standard integral. We can do some suitable change of variables or use `value` directly since this kind of substitution is very straightforward.

\[
> \text{value}(\%);
\]

There is another way to solve the above example question. The command `powsubs` allows us to eliminate even powers of \( \sin() \) or \( \cos() \) using \( \sin(x)^2 = 1 - \cos(x)^2 \) or \( \cos(x)^2 = 1 - \sin(x)^2 \).

\[
> \text{with(student)}:\powsubs(\sin(x)^2 = 1-\cos(x)^2,\text{Inte});
\]

\[
> \text{changevar}(v=\cos(x),\%,v);
\]

\[
> \text{value}(\%);
\]

Finally, we re-express the answer in term of \( x \).

\[
> \text{subs}(v=\cos(x),\%);
\]

Now, this answer looks different from the one we got before. Can you show that these two answers differ by at most a constant?

Example 2. Evaluate \( \int \frac{\sin(x)^5}{\cos(x)^{(2/3)}} \, dx \).

Here \( m = 5 \) is an odd number, so we use the substitution \( \cos(x) = t \), then \( \sin(x) \, dx = -dt \).

\[
> \text{Inte}:=\text{Int}(\sin(x)^5/(\cos(x)^{(2/3)}),x);
\]

\[
> \text{changevar}(t=\cos(x),\%,t);
\]

\[
> \text{expand}(\%);
\]

\[
> \text{value}(\%);
\]

\[
> \text{subs}(t=\cos(x),\%);
\]

Thus \( \int \frac{\sin(x)^5}{\cos(x)^{(2/3)}} \, dx = \)
\[-3 \cos(x)^4 + \frac{6 \cos(x)^2}{7} - \frac{3 \cos(x)}{13} + C.\]

**Example 3.** Evaluate \(\int \tan(x)^3 \sec(x)^4 \, dx\).

Since \(\tan(x) = \frac{\sin(x)}{\cos(x)}\) and \(\sec(x) = \frac{1}{\cos(x)}\), we can rewrite the integral in terms of sine and cosine functions.

\[
> \text{Inte}:= \text{Int}(\tan(x)^3 \sec(x)^4, x);
\]

\[
> \text{subs}(\tan(x) = \sin(x)/\cos(x), \sec(x) = 1/\cos(x), %);
\]

Here \(m + n = 10\) is even, so we can use the substitution \(\tan(x) = t\), then \(\frac{1}{\cos(x)^2} = 1 + t^2\) and \(\frac{dx}{\cos(x)^2} = dt\).

\[
> \text{changevar}(t = \tan(x), %, t);
\]

\[
> \text{expand}(%);
\]

\[
> \text{value}(%);
\]

\[
> \text{subs}(t = \tan(x), %);
\]

Thus \(\int \tan(x)^3 \sec(x)^4 \, dx = \frac{\tan(x)^4}{4} + \frac{\tan(x)^6}{6} + C\).

The second type of integral is of the form \(\int R(\sin(x), \cos(x)) \, dx\) where \(R\) is a rational function of \(\sin(x)\) and \(\cos(x)\). The universal substitution \(\tan(\frac{x}{2}) = t\) can always transform \(\int R(\sin(x), \cos(x)) \, dx\) into a rational integral. In this case, we need to memorize some trigonometric identities such as \(\sin(x) = \frac{2 \tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}\), \(\cos(x) = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}\). If we take \(\tan(\frac{x}{2}) = t\), then \(\sin(x) = 2 \arctan(t)\), thus \(dx = \frac{2 \, dt}{1 + t^2}\). Now we will try some examples of this type.

**Example 4.** Find \(\int \frac{1}{\sin(x)(2 + \cos(x) - 2 \sin(x))} \, dx\).

\[
> \text{Inte}:= \text{Int}(1/(\sin(x)*(2+\cos(x)-2*\sin(x))), x);
\]

\[
> \text{changevar}(t = \tan(x/2), %, t);
\]

\[
> \text{expand}(%);
\]

\[
> \text{value}(%);
\]

Thus we reformulate the original trigonometric integral to a rational integral. Then we can apply the techniques in the previous lab to find the final answer.

2. Integration of Hyperbolic Functions

Since hyperbolic functions have some identities which are similar to those of trigonometric functions, the integration of hyperbolic functions can be done in the same way as trigonometric functions. For integral of type \(\int R(\sinh(x), \cosh(x)) \, dx\), normally we will try the substitution \(\sinh(x) = t\), or \(\cosh(x) = t\), or \(\tanh(x) = t\) to reformulate the integral to a rational integral.

**Example 5.** Evaluate \(\int \frac{\cosh(x)^3}{\sinh(x)(1 - \cosh(x))} \, dx\).

\[
> \text{Inte}:= \text{Int}(\cosh(x)^3/(\sinh(x)*(1-\cosh(x))), x);
\]

\[
> \text{changevar}(t = \sinh(x), %, t);
\]

\[
> \text{changevar}(u = \sqrt{1 + t^2}, %, u);
\]
3. Square Roots of Quadratics

The technique used to find integrals of type \( \int R(x, \sqrt{ax^2 + bx + c}) \, dx \) is very closely related to the integration of trigonometric and hyperbolic functions. The trigonometric identities \( \sin(x)^2 = 1 - \cos(x)^2 \), \( \cos(x)^2 = 1 - \sin(x)^2 \), \( \tan(x)^2 = \sec(x)^2 - 1 \) and \( \sec(x)^2 = \tan(x)^2 + 1 \) can be used to simplify integrands involving \( \sqrt{1 - x^2}, \sqrt{x^2 - 1} \) and \( \sqrt{1 + x^2} \). Also we can apply the hyperbolic identities since they are similar. A general strategy is

1. Complete the square under the square root.
2. Perform a change of variables, followed by a removal of common factors, in order to rewrite the completed square as \( 1 + u^2 \) or \( 1 - u^2 \).
3. Use one of the above identities to turn the expression into a perfect square.

Following these steps will transform the original integral to an integral of trigonometric functions (or hyperbolic functions if we apply the hyperbolic identities).

**Example 6.** Evaluate \( \int \frac{x}{\sqrt{3 - 2x - x^2}} \, dx \).

\[
> \text{Inte} := \text{Int}(x / \text{sqrt}(3 - 2*x - x^2), x);
> \text{completesquare}(%), x);
> \text{changevar}(u = x + 1, %, u);
> \text{changevar}(u = 2*v, %, v);
> \text{transform the expression to involve}
> \sqrt{1 - v^2}
> \text{normal}(%) ;
> \text{simplify}(%);
> \text{changevar}(v = \sin(z), %, z);
> \text{simplify}(%);
> \text{value}(%);
\]

In the answer there is a sign function “\( \text{csgn}(\cos(z)) \)”. The value of the sign function is the sign of the variable. In this case, \( \text{csgn}(\cos(z)) = 1 \) if \( 0 < \cos(z) \), \( \text{csgn}(\cos(z)) = -1 \) if \( \cos(z) < 0 \). For the details, please try \( \text{?csgn} \). Also notice that the above answer is not the final answer for the original integral. We need to do all the back-substitutions in order to write the answer in terms of \( x \).

\[
> \text{?csgn}
\]

4. Reduction Formulas

We will conclude this Lab with an example of finding a reduction formula with the help of Maple.

**Example 7.** Find a reduction formula for \( \int \sin(x)^n e^x \, dx \), and use it to find \( \int \sin(x)^5 e^x \, dx \), i.e. \( n = 5 \).

Since the integral depends on the parameter \( n \), we will write it as a function of \( n \).

\[
> \text{e1} := n -> \text{Int}((\sin(x))^n * \exp(x), x);
\]
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> intparts(e1(n), sin(x)^n);
> integrate by parts with
> \( U = \sin(x)^n \)
> intparts(\%, sin(x)^(n-1)*cos(x));
> integrate by parts a second time

Now, we try to put the integrals into a form similar to \( e_1(n) \). This part requires practise. We made several tries and found that the following commands worked.

> subs(cos(x)^2=1-sin(x)^2,\%);
> expand(\%);
> simplify(\%, power);

Do you recognize the \( e_1(n) \) and the \( e_1(n-2) \) terms in the above answer? We will assign them to be another function so that later on we can solve them to get the reduction formula.

> subs(e1(n)=s(n), e1(n-2)=s(n-2),\%);
> solve(s(n)=\%, s(n));
> collect(\%, s(n-2));
> s:=unapply(\%, n);

The reduction formula expresses \( s(n) \) in terms of \( s(n-2) \), we need two values \( s(0) \) and \( s(1) \) to compute \( s(n) \) for larger integers \( n \).

> s(0):=value(e1(0));
> s(1):=value(e1(1));

Maple can work with the reduction formula. Now if we ask it for \( s(n) \) where \( n \) is a positive integer, Maple will use the reduction formula until it comes to \( s(0) \) or \( s(1) \), and then use the values of \( s(0) \) and \( s(1) \) we just entered. Let’s try to evaluate \( s(5) \).

> s(5);

5. Exercises

• 1. Evaluate the following integrals:
   a) \( \int \frac{\cos(x)^3}{\sin(x)} \, dx \);
   b) \( \int \frac{1}{5+\sin(x)+3\cos(x)} \, dx \);
   c) \( \int \frac{\sin(x)^2}{\cos(x)} \, dx \);
   d) \( \int \frac{2\tan(x)+3}{\sin(x)^2+2\cos(x)^2} \, dx \).

• 2. Find \( \int \frac{\sin(3x)}{1-\cos(x)} \, dx \).

• 3. Find \( \int \sqrt{\sec(x)^2 + a^2} \, dx \).

• 4. Find \( \int \sqrt{\tan(x)} \, dx \).
• 5. Identify and make use of standard trigonometric identities to carry out the transformation represented by

\[ \text{combine}(\sin(x)^3 \cos(x)^3, \text{trig}); \]

• 6. Use the commands \texttt{powsubs()} and \texttt{changevar()} to reexpress the integral \( \int \sin(x)^3 \cos(x)^3 \, dx \) as the integral of a polynomial.

• 7. Use the command \texttt{combine(..., trig)} to convert the integrals \( \int \sin(x)^8 \, dx \) and \( \int \cos(x)^8 \, dx \) into a form that can easily be integrated knowing only a value for \( \int \cos(u) \, du \) and the change of variable technique.

• 8. Use substitutions to reformulate

\[ \int \tan(x)^4 \sec(x)^6 \, dx \quad \text{and} \quad \int \tan(x)^3 \sec(x)^5 \, dx \]

as integrals of polynomials.

• 9. Reformulate the integral \( \int \sqrt{16 - x^2} \, dx \) as an integral involving \( \sqrt{1 - u^2} \) for some suitable choice of \( u \).

• 10. Find the indefinite integral for the following functions:

a) \( \frac{x^{(-2)}}{\sqrt{x^2 + 4}} \); b) \( \frac{3x}{\sqrt{8 - 2x - x^2}} \).

• 11. Find the reduction formulas for the following general integrals:

a) \( a_m = \int \frac{x^n}{\sqrt{x^2 + 1}} \, dx \); b) \( a_n = \int x^n \arctan(x) \, dx \).

**Lab 27. Introduction to Definite Integral**

In this lab, we will use Maple as an aid to understand the concept of the definite integral as a limit of Riemann sums.

1. Summations
First, we need to be familiar with the following powerful notation for summations. Let $x_1, x_2, \ldots, x_k$ be a finite sequence of $k$ $x$ values, then $\sum_{i=1}^{n} f(x) = f(x_1) + f(x_2) + \ldots + f(x_k)$.

The $\sum$ indicates our intention to compute a sum. The term $f(x_i)$ is a typical summand, and it must assume a value for each value of $i$. The notation must specify the following: the variable name $i$ that is to range over a set of integer values; a lower bound for $i$, which is the first value of $i$ in the sequence of summands; and an upper bound for $i$, which is the last value for $i$ in the sequence of summands.

Maple performs sums using the $\text{sum}$ command. For example, Maple’s command $\text{sum}(expr,i=m..n)$ means $\sum_{i=m}^{n} expr$, where $expr$ is an expression involving the variable $i$. Be careful that no value is assigned to the index variable $i$ before executing this $\text{sum}$ command.

```maple
> i:='i';
> sum(sin(i),i=2..4);
```

If Maple can evaluate the sum in closed form, it will do so. Otherwise it just returns the sum using sigma notation.

```maple
> sum(i^2,i=1..n);
> sum(1/sin(k),k=1..m);
```

The $\text{Sum}$ command is an inert form of $\text{sum}$, similar to $\text{Int}$ of $\text{int}$. We can use it to produce a sum in sigma notation without evaluating it.

```maple
> Sum(i^2,i=1..n);
```

The reason we might want to do this is to manipulate the sum in various ways. For example, we could try the $\text{combine}$ and $\text{expand}$ commands. To do this, we will first access the $\text{student}$ package.

```maple
> with(student):
> combine(a*Sum(i^2,i=1..n)+Sum(i^3,i=1..n));
> expand(%);
```

We could also use the $\text{changevar}$ command to change the index variable. However, the new variable should not already be present in the expression.

```maple
> changevar(i=j+1,%,j);
```

Again, similar to the integral, we can use the $\text{value}$ command if we want to evaluate a $\text{Sum}$.

```maple
> value(%);
```

2. The Riemann Sums

Suppose we divide the interval $[a, b]$ into $n$ equal subintervals. Then the Riemann sum can be written as $\sum \frac{f(\xi_k) (b-a)}{n}$, where the points $\xi_k$ in $[x_{k-1}, x_k]$ are chosen arbitrarily. However, we could try it in three different ways: $\xi_k = x_{k-1}$, $\xi_k = \frac{x_{k-1} + x_k}{2}$ and $\xi_k = x_k$. In Maple, the $\text{student}$ package has the corresponding commands: $\text{leftsum}$, $\text{middlesum}$ and $\text{rightsum}$. 
Appendixes:

Geometrically, all these are rectangle approximations of the area of the region under the curve \( y = f(x) \).

Now, take an example \( y = \sin(x) \) with \( a=0 \), \( b=2 \) and \( n=10 \). What are the sums?

\[
\text{leftsum}(\sin(x), x=0..2, 10); \\
\text{value}(\%); \\
\text{evalf}(\%); \\
\text{value}(\text{middlesum}(\sin(x), x=0..2, 10)); \\
\text{evalf}(\%); \\
\text{evalf}(\text{rightsum}(\sin(x), x=0..2, 10));
\]

The decimal answer for these leftsum, middlesum and rightsum are not very close together. Let us try a larger \( n \).

\[
\text{evalf}(\text{leftsum}(\sin(x), x=0..2, 100)); \\
\text{evalf}(\text{middlesum}(\sin(x), x=0..2, 100)); \\
\text{evalf}(\text{rightsum}(\sin(x), x=0..2, 100));
\]

You could try even larger and larger \( n \) to see what happens. But here, let us try to see what will happen as \( n \to \infty \).

\[
\text{value}(\text{leftsum}(\sin(x), x=0..2, n)); \\
\text{limit}(\%, n=\text{infinity}); \\
\text{evalf}(\%); \\
\text{value}(\text{middlesum}(\sin(x), x=0..2, n)); \\
\text{limit}(\%, n=\text{infinity}); \\
\text{value}(\text{rightsum}(\sin(x), x=0..2, n)); \\
\text{limit}(\%, n=\text{infinity});
\]

3. Definite Integrals

We see that as \( n \to \infty \), the values of the limit for all three sums coincide. This limit is the definite integral of \( \sin(x) \) on the interval \([0, 2]\). Maple can calculate the definite integral directly.

\[
\text{int}(\sin(x), x=0..2);
\]

Also we can evaluate the definite integral ourselves, using an antiderivative of \( \sin(x) \) and the Fundamental Theorem of Calculus (coming in the next lab).

\[
\text{int}(\sin(x), x); \\
\text{subs}(x=2, \%)-\text{subs}(x=0, \%);
\]

Notice that \( 2 - \cos(1)^2 = 1 - \cos(2) \) since \( 2\cos(1)^2 - 1 = \cos(2) \).

4. Exercises
1. Evaluate the following using \texttt{sum}.
   a). \(1^2 + 2^2 + \ldots + 10^2\).
   b). \(\sum_{i=1}^{5} \frac{2i+1}{2i-1}\).
   c). \(a_1 b_1 + a_4 b_2 + a_3 b_3 + a_2 b_4 + a_1 b_5\) where \(a_j = 2j + 1\) and \(b_j = 2j - 1\).

2. In using \texttt{changevar} on \(s = \sum_{i=a}^{b} f(i)\), there must be a one-to-one correspondence between the values of the original index and the integer values of the new index variable in some integral. Which of the following are valid uses of \texttt{changevar}?
   a). \texttt{changevar}( i = j + 1, s, j);
   b). \texttt{changevar}( i = j^2, s, j);
   c). \texttt{changevar}( j = b - i, s, j);

3. a). Using Maple show that \(\sec((n+1)a) \sec(na) = \frac{\tan((n+1)a) - \tan(na)}{\sin(a)}\).
   b). What does this imply about \(\sum_{j=1}^{n} \sec((j+1)a) \sec(na)\)? Can Maple evaluate this sum?
   c). Verify the sum for \(n = 5\) using Maple.

4. Calculate the left, middle and right sums for the following functions on the interval \([a, b]\). Find the limit as \(n \to \infty\), and compare it to \(\int_{a}^{b} f(x) \, dx\).
   a). \(f(x) = x^2\).
   b). \(f(x) = x^5\).
   c). \(f(x) = x^u\).
   d). \(f(x) = \cos(x)\).

5. Consider the function \(f(x) = x^3 - 2x^2 - 3x + 10\) on the interval \([-2, 3]\).
   a). Calculate numerically the left, middle and right sums for \(n = 10\) and \(n = 100\).
   b). Find the limit of the left sum as \(n \to \infty\). Compare this to the area as found using \texttt{int}.
   c). Plot the graph of \(f\) and rectangles for its left and right sums for \(n = 20\) (in one graph). Hint: use Maple commands \texttt{leftbox}, \texttt{rightbox}.
6. Consider the function $f(x) = x + \cos(x)$ on the interval $[-1, 2]$.
   a). Of the left, middle and right sums for any given $n$, which are the same?
   b). What is the least $n$ for which the difference between the left sum and the right sum is less than 0.01 without knowing the values of the sums?
   c). For $n = 100$, how close are the left, middle, right sums to the definite integral? Can you conclude any practical usefulness of these sums as approximations of the area?

7. Let $L_n$, $R_n$, and $M_n$ be the left, right, middle sums, respectively for $f(x) = x^2$ on $[a,b]$ with a partition into $n$ equal intervals. Find $\lim_{n \to \infty} n \left( L_n - \int_a^b f(x) \, dx \right)$ and the limits with $L_n$ replaced by $R_n$ and by $M_n$. Try it with some other $f$ for which Maple can find formulas for $L_n$, $R_n$ and $M_n$. Does this help to explain why $M_n$ is a better approximation to the definite integral?

---

### The Fundamental Theorem of Calculus

In this Lab, we will explore the fact that the integration and differentiation are, in some sense, inverse operations.

1. **The Fundamental Theorem**
   We have seen definite integration in the previous Lab. Now, we fix the lower end point of the definite integral and allow the other end point to vary. Each choice of the upper endpoint yields a real number, so by varying $x$, we obtain a function.

   ```maple
   > A:=x->int(f(t),t=a..x);
   > A(d)-A(c);
   > combine(%);
   ```

   The above answer tells us that $\int_c^d f(t) \, dt = A(d) - A(c)$. In fact, the function $A$ is often called an antiderivative of $f$. The terminology antiderivative comes from the result: $D(A)(x) = f(x)$ if $A(x) = \int_a^x f(t) \, dt$. We can prove this with Maple as an assistant to do all the algebraic manipulation.

   ```maple
   > (A(x+h)-A(x))/h;
   > the Newton quotient
   > num:=combine(A(x+h)-A(x));
   ```

   Suppose $h > 0$. Since $f(x)$ is bounded on the region $[x, x + h]$, we have upper and lower rectangles of height $m$ and $M$ that bound the function $f$ from below and above on this interval. These $m$ and $M$ can be chosen as $m = f(u)$ and $M = f(v)$, where $u$ and $v$ are in $[x, x + h]$. Then we have:
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> r1:=Int(m,t=x..x+h)<=Int(f(t),t=x..x+h);
> r2:=Int(f(t),t=x..x+h)<=Int(M,t=x..x+h);

So
> e1:=value(subs(m=f(u),r1/h));
> e2:=value(subs(M=f(v),r2/h));

Now as \( h \to 0 \), \( f(u) \) and \( f(v) \) tend to \( f(x) \), and the derivative of \( A \) at \( x \) is bounded by \( f(u) \) and \( f(v) \).
> limit(lhs(e1),h=0)<=Limit(rhs(e1),h=0);  
> Limit(lhs(e2),h=0)<=limit(rhs(e2),h=0); 

As \( f(u) \) and \( f(v) \) become equal to \( f(x) \) in the limit, this proves \( f(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t) \, dt}{h} \).

Next, let’s take a look at the graphs of the function \( f \), the antiderivative \( A \) and the Newton quotient \( NQ \) for a specified function. Consider the function \( f \) defined by \( f(x) = \sqrt{x^2+2} \) on the interval \([1, 3]\) and the Newton quotient at \( x \) for the antiderivative with \( h = .2 \).
> f:=x->sqrt(x^2+2);
> the function
> f
> A:=x->int(f(t),t=a..x);
> the antiderivative of
> f
> NQ:=x->1/h*int(f(t),t=x..x+h);
> the Newton quotient of
> A at
> x

We plot these \( f, A \) and \( NQ \) simultaneously on the interval \([1, 3]\).
> a:=1; h:=0.2;
> plot({A,NQ,f},1..3);

Can you tell which curve is for which function? With some manipulation (e.g. add an option in \texttt{plot} ), we can see that the curves for \( NQ \) and for \( f \) are almost parallel and are very close to each other.
> evalf(NQ(2)), evalf(f(2)), evalf(A(2));

With the fundamental theorem of calculus, the calculation of definite integrals becomes easy since we already know how to find an antiderivative of \( f \), i.e. the infinite integral of \( f \). For instance, \( \int_a^b x^2 \, dx = \frac{b^3}{3} + \frac{a^3}{3} \) since \( \int x^2 \, dx = \frac{x^3}{3} + C \); \( \int_a^b \cos(x) \, dx = \sin(b) - \sin(a) \) since \( \int \cos(x) \, dx = \sin(x) + C \). Generally speaking, \( \int_a^b f(x) \, dx = A(b) - A(a) \) if \( \int f(x) \, dx = A(x) + C \). However, this doesn’t apply in general. We will discuss some special cases in the next section.

2. Application of the Fundamental Theorem

We can use the fundamental theorem to evaluate a definite integral. However, before doing that, make sure that the function \( f \) is continuous.
Consider the problem of finding the area under the curve defined by \( f(x) = \frac{1}{x^2} \) on the interval \([-1, 1]\).

```maple
with(student):
f:=x->1/x^2;
int(f(x),x);
A:=makeproc(%,x);
A(1)-A(-1);
```

If we try the fundamental theorem directly, we get the answer -2.

```maple
int(f(x),x=-1..1);
```

If we use the definite integral function from Maple, it gives the answer \( \infty \). Which one is right? What’s wrong with the other one? Let’s plot the graph of this function.

```maple
plot(f,-1..1,0..10);
```

There is a vertical asymptote at \( x = 0 \). The function \( f \) is not continuous on the interval \([-1, 1]\). We can use `iscont` to check for continuity.

```maple
readlib(iscont):
command iscont is in the package iscont
iscont(f,x=-1..1);
```

For a continuous function \( f \), the fundamental theorem is very useful. Assume the velocity of a given particle is given by the function \( v(t) = t^2 - 8t + 5 \), where the time \( t \) is in seconds and \( v \) is in meters per second. Find a formula for the position of the particle as a function of time. Use this formula to find the total displacement of the particle from the time \( t = 1 \) to time \( t = 5 \).

First, define the velocity function \( v(t) \).

```maple
v:=t->t^2-8*t+5;
```

We know that velocity is the derivative of the position vector as a function of time, so the desired position function is one of the antiderivatives of \( v \).

```maple
Int(v(t),t)+C;
value(%);
d:=makeproc(%t);
```

Remember to add a constant \( C \) for the antiderivative. The constant \( C \) is arbitrary. It tells us that the final answer to the first question is not unique.

For the second question, we are asked for displacement. By comparing the two positions of the particle, we can compute the difference in positions, i.e. the total displacement.

```maple
d(5);
end position

d(1);
start position

d(5)-d(1);
the total displacement
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Notice that the answer for the second question is unique. We can also check this by using Maple’s command for definite integral.

> int(v(t), t=1..5);

3. Exercises

- 1. Use Maple to find the indefinite integrals of each of the following, and then differentiate the answers. Compare the result in each case to the original function. Are they always the same? *Hint:* It may help to apply the command `simplify` to your results.
  a). $x \sqrt{1 + x^2}$
  b). $x \tan(x^2)$
  c). $\ln\left(\frac{1+x}{x}\right)$
  d). $\sin(x^2)$
  e). $\sin(\cos(x^2))\sin(x^2)\cdot x$

- 2. Express the area under the curve $f$ defined by the function $f(x) = x^3 - 9x^2 + 26x - 24$ from $x = 2$ to $x = 3$ in terms of integrals, and find its value.

- 3. Express the area between the curve $f$ defined by the function $f(x) = x^3 - 9x^2 + 26x - 24$ and the $x$-axis, from $x = 1$ to $x = \frac{5}{2}$, in terms of integrals and find its value.

- 4. Find the exact area between the curves $f$ and $g$ defined by the functions $f(x) = x^3 - 9x^2 + 26x - 24$ and $g(x) = -x^3 + 9x^2 - 26x + 22$ from $x = 2$ to $x = 4$, and from $x = 1$ to $x = 4$. *Hint:* Plot the graphs of these functions.

- 5. Consider the function $f(x) = \sqrt{3x^2 + 5}$. Define and plot an antiderivative of $f$ on the interval $[2, 4]$ and the Newton quotient for the antiderivative with $h = .2$. Experiment with different values of $h$ to discover how the size of $h$ affects the accuracy of the approximation.

- 6. The velocity of a particle is given by the function $v(t) = t^2 - 10t + 14$. Find a formula for the position of the particle as a function of time $t$. Is it unique? Also find the total displacement of the particle from time $t = 1$ to time $t = 5$ from the position formula you just derived. Check your answer with Maple’s definite integral command.
• 7. Find the total displacement of a particle from \( t = 0 \) to \( t = 10 \) if its velocity at time \( t \) is given by \( v(t) = \frac{e^{-t^3}}{t+1} \).

**Lab 29. Numerical Integration**

Sometimes it’s impossible to find an antiderivative of \( f(x) \), and in this case we can’t evaluate the definite integral \( \int_a^b f(x) \, dx \) exactly by applying the fundamental theorem of calculus. However, we can find an approximate value of the definite integral \( \int_a^b f(x) \, dx \). There are several methods of approximation for a definite integral. In this Lab, we study some methods to numerically approximate integrals, and about how to estimate and control the errors in these approximations.

1. **Introduction**

Before we try any numerical approximation method, we need to understand what Maple can do. Maple can find the exact values for antiderivatives and definite integrals if they exist. In addition to this, Maple contains very sophisticated methods for finding numerical approximations to definite integrals. Command `int` asks Maple to find the exact value. If there is no closed form for the integral, Maple will return the integral expression without doing anything. Command `evalf(expr)` asks Maple to find the numerical value. This `evalf` requires all the variables in `expr` to be specified as numerical values. Can you tell the difference between the following two commands?

\[
\text{evalf(int(f(x), x=a..b));}
\]
\[
\text{evalf(Int(f(x), x=a..b));}
\]

The next three sections of this lab will talk about some methods of numerical integration: the rectangular rule, the trapezoid rule and Simpson’s rule. These are less sophisticated than Maple’s method, but easier to understand. It’s good for us to understand how the methods work. However, for practical calculation, `evalf(Int(..))` is almost always be much better.

We will use many commands from `student` package, which we should load before doing any work in this lab.

\[
\text{with(student);} \]

2. **Rectangular Methods**

We already have seen some parts of this methods (leftsum, middlesum, rightsum) in previous labs. From the following example, we can understand more about the techniques and the corresponding errors.

Consider the function \( f(x) = x^5 - 5x^2 \). Before we do any algebraic or numerical computation, let’s take a look at the graphs.

\[
\text{f:=x->x^5-5*x^2;} \]
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From the graphs, can you tell what area is associated with \( f^1_0 f(x) \, dx \), for leftsum \( f(x), x = 0..1, 6 \), for rightsum \( f(x), x = 0..1, 6 \), and for middlesum \( f(x), x = 0..1, 6 \)?

Next, we define \( A \) as the true definite integral, \( M(n) \), \( L(n) \), and \( R(n) \) the middle, left and right sums as functions of \( n \), the number of sub intervals.

The errors in the approximations are defined as following:

The answers to the above say that as \( n \to \infty \), the sums approach the definite integral, which is the theoretical reason behind us to use these sums as the numerical approximations.

Now, we can look at these functions for particular values of \( a, b \) and \( n \). But it’s more interesting to see how they behave for very large \( n \).

The first two limits of the above are as some functions of \( a \) and \( b \), i.e. nonzero number \( C \). They tell us that \( LE(n) \) and \( RE(n) \) is approximately \( \frac{C}{n} \) for large \( n \). The last limit is zero, which means that \( ME(n) \) goes to 0 faster than any \( \frac{C}{n^2} \). Then what about \( \frac{C}{n^2} \)?

This confirms that \( ME(n) \) is approximately \( \frac{C}{n^2} \) for large \( n \). What is the constant \( C \)?

In fact, \( C = \frac{(Df(b)-Df(a)) \cdot (b-a)^2}{24} \).

This answer should be 0, which means the average of the left and the right sums is a better approximation than either of them. We will discuss this in the next section.

3. Trapezoidal Method

In fact, the average of the left and right sum has a name: the trapezoid rule. The student package has it as trapezoid.
Appendixes:

Let’s check the fact that the trapezoid rule approximation $T(n)$ is the average of $L(n)$ and $R(n)$.

```
> trapezoid(g(x),x=a..b,n);
```

Now, we define $T(n)$ and its error $TE(n)$ similarly to $L(n)$, $R(n)$ and $M(n)$.

```
> T:=unapply(value(trapezoid(f(x),x=a..b, n)),n);
> TE:=unapply(A-T(n),n);
```

How does the error behave for very large $n$?

```
> limit(n^2*TE(n),n=infinity);
```

The answer means that $TE(n)$ behaves similar to $ME(n)$.

4. Simpson’s Method

Simpson’s rule is better than the previous two methods. The student package has a command `simpson` for it.

```
> simpson(g(x),x=a..b,n);
```

Let’s try it for our example function $f(x) = x^5 - 5x^2$.

```
> value(simpson(f(x),x=a..b,n));
> expand(%);
> int(f(x),x=a..b)-%;
```

This is the error in Simpson’s rule for our function. It is of the form $C \frac{n^4}{4}$. What is the constant $C$?

```
> simplify(%*n^4-((D@@3)(f)(a)-(D@@3)(f)(b)) *(b-a)^4/180);
```

For this function $f(x)$, the error $SE(n)$ is approximately $C \frac{n^4}{4}$ with $C = \frac{(b^4)(f(a) - (D@@3)(f)(b))^2}{180}$. This means that Simpson’s rule should have much smaller error than the trapezoid or any rectangular method when $n$ is large.

Please notice the fact that Simpson’s rule can be expressed as a combination of the trapezoid rule and the midpoint rule. In fact, $\frac{2M(n)}{3} + \frac{T(n)}{3}$ is Simpson’s approximation.

5. Exercises

- 1. Suppose $a = -1$ and $b = 2$ for our example function $f(x) = x^5 - 5x^2$.

```
> a:=-1;
> b:=2;
> limit(n*LE(n),n=infinity);
```

Which means $LE(n)$ is almost $\frac{27}{n}$ for large $n$. By this, if we want the accuracy to be $10^{-9}$, we’d need $n$ to be almost $\frac{27}{10^{-9}}$, i.e., $2.7 \times 10^{10}$. 

```
Appendix 2: Introduction to Maple V Labs

Now consider the integral \( \int_{0}^{1} \frac{1}{1+x^2} \, dx = \frac{\pi}{4} \). If we want to estimate the value of \( \pi \) with accuracy 0.001, what should \( n \) be if we apply the three approximation methods for the definite integral: rectangular methods, trapezoid method and Simpson’s method?

2. Let \( f(x) = \frac{1}{1+x} \).
   a). Approximate \( \int_{0}^{1} f(x) \, dx \) using the midpoint, trapezoid and Simpson’s rules for \( n = 10 \). Find the error in each approximation.
   b). Calculate the limits as \( n \to \infty \) of \( n^2 \) times the error in Simpson’s rule.
   c). How large must \( n \) be so that the absolute value of the error in the midpoint rule is less than 0.0001? Do the same for the trapezoid rule and Simpson’s rule.

3. Consider the integral \( \int_{-1}^{2} \sqrt{1+x^3} \, dx \). Let \( ME(n) \), \( TE(n) \) and \( SE(n) \) be the errors in the midpoint, trapezoid and Simpson’s rules for this integral.
   a). Find \( ME(n) \), \( TE(n) \) and \( SE(n) \) explicitly as functions of \( n \).
   b). Plot \( n^2 ME(n) \) and \( n^2 TE(n) \) (on the same graph) and \( n^4 SE(n) \) (on a different graph) as functions of \( n \). What can you conclude?
   c). Find the limits of \( n^2 ME(n) \), \( n^2 TE(n) \) and \( n^4 SE(n) \) as \( n \to \infty \).
   d). What \( n \)'s would be needed for the errors in the midpoint, trapezoid and Simpson’s rules to be less than \( 10^{(-9)} \)?

4. Consider \( \int_{0}^{1} \sqrt{x} \, dx \). Let \( ME(n) \), \( TE(n) \) and \( SE(n) \) be the errors in the midpoint, trapezoid and Simpson’s rules for this integral.
   a). Plot \( n^{(\frac{3}{2})} ME(n) \), \( n^{(\frac{3}{2})} TE(n) \) and (for even \( n \) \( n^{(\frac{3}{2})} SE(n) \) (on the same graph) as functions of \( n \). What can you conclude?
   b). Approximately what \( n \) would be needed for the error in Simpson’s rule to be less than \( 10^{(-9)} \)?

5. a). Let \( f(x) = x^{11} \). Find the error \( TE(n) \) in the trapezoid rule for the interval \([a, b]\). Show that this is of the form \( \sum_{j=1}^{5} c_j (b-a)^{(2j-1)} ((D^{(2j-1)}(f))(b) - (D^{(2j-1)}(f))(a)) \), where \( c_j \) are constants (not involving \( a \) and \( b \)). Hint: to find the coefficient of \( n^{(-2j)} \), use \text{coeff}(TE(n),n,-2*j) \).
   b). Now try it for other power \( x^m \) \((5 \leq m)\). Show that \( TE(n) \) has the same form. It turns out that for any function with at least \( 2m+2 \) derivative on \([a, b]\),
Appendixes:

\[ \text{TE}(n) = \left( \sum_{j=1}^{m} c_j \frac{(b-a)^{(2j)}}{n^{(2j)}} \frac{f(b) - (D(2j-1))f(a)}{j} \right) + O\left(n^{-2m-2}\right), \]

where \(c_j\) are constants that don’t depend on \(n, f, a\) or \(b\) (and \(c_1\) to \(c_5\) are the same as in part a)).

\[ c_j = \begin{cases} \text{constant} & \text{if } j \leq 5, \\ 0 & \text{otherwise} \end{cases} \]

\(c_j\) are constants that don’t depend on \(n, f, a\) or \(b\) (and \(c_1\) to \(c_5\) are the same as in part a)).

\[ \text{c). Using } f(x) = e^x, a = 0, b = 1, \text{ compare the actual } \text{TE}(n) \text{ to the sum from } j = 1 \text{ to } 5 \text{ in the above formula. Calculate } n^{12} \text{ times the difference, for } n = 1 \text{ to } n = 10. \text{ Does this appear to approach the limit? Use } \text{Digits:=}30. \]

\[ \text{b). How accurate are the midpoint, trapezoid and Simpson’s rule approximations for } n = 20? \text{ (make } \text{Digits at least} 20) \text{ Do you find the result surprising?} \]

---

**Lab 30. Improper Integrals**

Riemann integration, shown as \( \int_{a}^{b} f(x) \, dx \), which we learned in this calculus course, is defined for bounded functions \( f(x) \) on a bounded interval \([a, b]\). Improper integrals are definite integrals over unbounded intervals or with unbounded integrands. These are extensions to the theory of integration. There are mainly two types of improper integrals, which will be fully discussed in this lab. Some texts mention a third type, which is a combination of the first two types. Thus, our basic task in this lab is to understand and compute the first two types of improper integral, by using the definitions from the textbook and using Maple’s commands directly.

1. **Improper Integral of the First Type**

   The first type refers to the case of definite integrals over unbounded intervals, of form \( \int_{-\infty}^{\infty} f(x) \, dx \), \( \int_{-a}^{a} f(x) \, dx \), or \( \int_{-\infty}^{a} f(x) \, dx \). The definitions of these improper integrals are in the textbook. Here the question is how to manipulate them in Maple.

   Consider a function defined by \( f(x) = \frac{1}{1+x^2} \).

   ```maple
   > f:=x->1/(1+x^2);
   > plot(f,-5..5);
   ```

   The function is shaped like a bell with an infinite “flare”. Find the area under this flare if \( x \) is in the interval \([1, \infty]\), i.e. find the value of the improper integral \( \int_{1}^{\infty} \frac{1}{1+x^2} \, dx \).

   According to the definition \( \int_{a}^{\infty} f(x) \, dx = \lim_{A \to \infty} \int_{a}^{A} f(x) \, dx \), we evaluate the integral by the following commands.

   ```maple
   > Int(f(x),x=1..A);
   > value(%);
   > Int1:=limit(% ,A=infinity);
   ```
Do the same thing to evaluate $\int_{-\infty}^{1} \frac{1}{1+x^2} \, dx$.

\[
\begin{align*}
> & \quad \text{Int}(f(x),x=A..1); \\
> & \quad \text{value}(\%); \\
> & \quad \text{Int2:=limit}(\%,A=-\text{infinity});
\end{align*}
\]

Now, it’s easy to find the value of $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$.

\[
\begin{align*}
> & \quad \text{Int1+Int2};
\end{align*}
\]

We can check all the three answers by using Maple commands directly.

\[
\begin{align*}
> & \quad \text{int}(f(x),x=1..\text{infinity}); \\
> & \quad \text{int}(f(x),x=-\text{infinity}..1); \\
> & \quad \text{int}(f(x),x=-\text{infinity}..\text{infinity});
\end{align*}
\]

Now, consider the parametric function defined by $f(t) = te^{-st}$ for $t$ on $[0, \infty]$, where $s$ is a positive number.

\[
\begin{align*}
> & \quad f:=t->t*\exp(-s*t);
\end{align*}
\]

Use Maple commands directly.

\[
\begin{align*}
> & \quad \text{int}(f(x),x=0..\text{infinity});
\end{align*}
\]

Maple can not give a closed form since it doesn’t know if $s$ is positive or negative. Hence, we need to tell Maple that we would like $s$ to be considered positive.

\[
\begin{align*}
> & \quad \text{assume}(s>0); \\
> & \quad \text{value}(\%);
\end{align*}
\]

The tilde ($\tilde{\,}$) attached to the $s$ reminds us that the value of the integral is $\frac{1}{s^2}$ only if the assumption is valid, i.e. $s$ is positive.

This improper integral is an example of a Laplace transform, which is very important and very useful in applied mathematics.

As we know from the previous Labs, some integrands don’t have explicit expressions for their antiderivatives.

Consider the function $f(x) = \frac{\cos(x)}{1+x^5}$ on $[0, \infty]$.

\[
\begin{align*}
> & \quad f:=x->\cos(x)/(1+x^5); \\
> & \quad \text{plot}(f(x),x=0..\text{Pi});
\end{align*}
\]

The graph shows that the oscillations from $\cos(x)$ in the numerator are rapidly damped out by the $1 + x^5$ in the denominator. Hence, integrating to a modest finite upper limit should yield an accurate approximation to the improper integral.

\[
\begin{align*}
> & \quad \text{evalf}(\text{Int}(f(x),x=0..10));
\end{align*}
\]

We could do more experiments to guess the answer for the improper integral, by choosing larger and larger upper limits for the numerical approximation. However, another kind of interesting question arises: can we prove the convergence of this improper integral. The answer is yes. Since $|\cos(x)| \leq 1$, we only need to study the convergence of the dominating integral $\int_{0}^{\infty} \frac{1}{1+x^5} \, dx$.

\[
\begin{align*}
> & \quad \text{Int}(1/(1+x^5),x=0..\text{infinity}); \\
> & \quad \text{value}(\%);
\end{align*}
\]
The dominating integral converges, so does the improper integral $\int_{0}^{\infty} \frac{\cos(x)}{1+x^5} \, dx$.

2. Improper Integral of the Second Type
This type refers to definite integrals with unbounded functions, of form $\int_{a}^{b} f(x) \, dx$, where $f(x)$ has a singular point at $x = a$, or at $x = b$, or at $x = c$ with $c$ in $[a, b]$.
Consider the function $f(x) = \frac{1}{\sqrt{4-x}}$ on the interval $[0, 4]$. The function has a singular point at $x = 4$. We compute the value of the improper integral $\int_{0}^{4} \frac{1}{\sqrt{4-x}} \, dx$ by using the definition.

$$\text{f:=x->1/sqrt(4-x);}
\text{int(f(x),x=0..4-epsilon);}
\text{limit(%,epsilon=0,right);}$$

Check the answer in Maple.

$$\text{int(f(x),x=0..4);}$$

The last example is related to the case when the singular point $c$ is in the interval $[a, b]$.
Evaluate the improper integral $\int_{-1}^{2} \frac{1}{x^3} \, dx$.
$\frac{1}{x^3}$ has a singular point at $x = 0$. We split the integral into two, the first over the interval $[-1, 0]$ and the second over the interval $[0, 2]$.

$$\text{f:=x->1/x^3;}$$
$$\text{int(f(x),x=0+epsilon..2);}
\text{Int1:=limit(%),epsilon=0,right);}$$
$$\text{int(f(x),x=-1..0-epsilon);}$$
$$\text{Int2:=limit(%),epsilon=0,right);}$$

What is $\infty - \infty$? It could be anything. We can try to obtain the Cauchy Principal Value (CPV), which is an attempt to balance the “positive area” against the “negative area” on either side of a singular point inside the integral interval.

$$\text{Int(f(x),x=-1..0-epsilon)+int(f(x), x=0+epsilon..2);}
\text{value(%)};$$
$$\text{limit(%,epsilon=0,right);}$$

Maple can obtain the CPV of an improper integral directly.

$$\text{int(f(x),x=-1..2,CauchyPrincipalValue);}$$

3. Exercises

• 1. Define $f(x) = x^{-\frac{3}{2}}$.
  a). Plot the graph of $f(x)$ for $x$ in $[1, 4]$.
  b). Find the area of the region in the first quadrant which is under $f(x)$ and between $x = 1$ and $x = 4$. 
c). Find the area of the region in the first quadrant which is under \( f(x) \) and to the right of \( x = 1 \).

\[ \int_{1}^{\infty} \frac{1}{e^{-x}} \, dx \]

\[ \int_{0}^{\infty} x \, e^{-x^2} \, dx \]

\[ \int_{1}^{10} \frac{x}{\sqrt{x-1}} \, dx \]

\[ \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} \, dx \]

\[ \int_{0}^{\infty} e^{(-x)} \, dx \]

\[ \int_{0}^{\infty} \frac{1}{4+x^2} \, dx \]

\[ \int_{0}^{\infty} \frac{1}{4+a^2} \, dx \]

\[ \int_{0}^{\infty} e^{(-a x)} \cos(b x) \, dx \]

for \( 0 < a \).
b). \[ \int_0^\infty e^{-ax} \sin(bx) \, dx \] for \( 0 < a \);

c). \[ \int_a^\infty \frac{1}{x^\lambda} \, dx \] for \( 0 < a \) and \( 1 < \lambda \).

Hint: Use Maple’s commands to compute the improper integrals to check your answer.

• 6. Assume that \( A(a) = \int_0^a t^3 e^{(-t)} \, dt \).

a). Compute the values of \( A(5) \), \( A(10) \), and \( A(20) \).

b). What is the smallest value of \( a \) such that \( A(a) \) is greater than 5.9995?

c). Can you make \( A(a) \) greater than 6 if you choose \( a \) large enough? Try several large values of \( a \) to see.

d). What can you guess about the value of \( \int_0^\infty t^3 e^{(-t)} \, dt \)? Prove your assertion by using the definition of improper integral.

• 7. Using hand calculations and integration by parts show that \( I(n) = n \, I(n-1) \), where \( I(n) = \int_0^\infty t^n e^{(-t)} \, dt \).

• 8. Calculate the induction formula for \( I(n) = \int_0^\infty t^n e^{(-t)} \, dt \). Compute the value of \( I(0) = \int_0^\infty e^{(-t)} \, dt \). Use your induction formula to compute \( I(3) \), \( I(8) \), \( I(20) \). Is this value of \( I(3) \) the same as your answer to question 6. d)?

Lab 31. Applications of Definite Integral (I)

This Lab mainly concentrates on computing the arc length of a curve and areas bounded by curves. We will learn to use Maple to plot parametric and polar curves. Then we will use the formulas derived in the textbook to compute the arc length and the areas.

1. Arc Length of a Curve

For a regular parametric curve \( \gamma: R \rightarrow R^2 \) with \( \gamma(t) = (\phi(t), \psi(t)) \), its length \( l(\gamma) = \int_a^b \sqrt{D(\phi)(t)^2 + D(\psi)(t)^2} \, dt \). The details of the proof are in theorem 7.1.2 of the textbook. Here we will give some examples to show how to plot a parametric curve, how to apply the above formula to compute the arc length, and how to re-write the curve in parametric form if the curve is expressed in a general equation form.

Example 1. Compute the length of the arc of the semicubical parabola \( y^2 = x^3 \) between the points \((0,0)\) and \((a, a^{\frac{3}{2}})\), where \( a \) is an arbitrary positive number.
This curve is expressed by a general equation in Cartesian coordinates. If we write
$y$ in terms of $x$, it includes two branches: $y = x^{(3/2)}$ and $y = -x^{(3/2)}$. We can see the
upper arc and the lower arc from the following graph. The command \texttt{implicitplot}

is in the \texttt{plots} package, which should be loaded before we try this command.

\begin{verbatim}
> eq:=y^2=x^3;
> with(plots):
> implicitplot(eq,x=0..4,y=-8..8);
\end{verbatim}

The question asks us to consider the arc between the origin (0,0) and $(a, a^{(3/2)})$, which
belongs to the upper branch of the graph. Therefore we can write the following
parametric equations: $x = t$ and $y = t^{(3/2)}$ for non-negative real numbers $t$. Then the
required arc length is obtained by changing $t$ from 0 to $a$.

\begin{verbatim}
> x:=t->t; y:=t->t^(3/2);
> l:=Int(sqrt((D(x)(t))^2+(D(y)(t))^2), t=0..a);
> value(%);
\end{verbatim}

Notice that in this lab and the next lab, we will skip all the details of definite integral
computation. However, for practice we should try to do all these computations by hand.

\textbf{Example 2.} Compute the length of the astroid $x^{(3/2)} + y^{(3/2)} = a^{(3/2)}$ in the first quadrant.
Again, this is the general equation in Cartesian coordinates. Drawing the graph may
help us understand more about the arc we need to consider.

\begin{verbatim}
> eq:=x^(2/3)+y^(2/3)=a^(2/3);
> a:=8;
> specify the value of
> a for plotting
> implicitplot(eq,x=0..9,y=0..9);
> a:='a';
> change
> a back to a general parameter
\end{verbatim}

We only plot the graph in the first quadrant. From the equation, we know that the
whole graph is symmetrical about the $x$-axis and the $y$-axis. In the first quadrant,
$0 < x$ and $0 < y$, we have $y = (a^{(3/2)} - x^{(3/2)})^{(3/2)}$ for $x$ in $[0, a]$. Thus we can represent
the arc $\gamma$ by the following parametric equations: $x = t$ and $y = (a^{(3/2)} - t^{(3/2)})^{(3/2)}$ for $t$
in $[0, a]$.

\begin{verbatim}
> x:=t->t; y:=t->(a^(2/3)-t^(2/3))^(3/2);
> l:=Int(sqrt((D(x)(t))^2+(D(y)(t))^2), t=0..a);
> value(%);
\end{verbatim}

Notice that here we actually compute an improper integral.

\textbf{Example 3.} Compute the length of the arc represented parametrically by the equations:
$x = a (\cos(t) + t \sin(t))$ and $y = a (\sin(t) - t \cos(t))$ for $t$ in $[0, 2 \pi]$.
This is a parametric curve. We can apply the arc length formula directly. Before
doing that, let’s take a look at the graph of the arc we are considering.
> x:=t->a*(cos(t)+t*sin(t));
> y:=t->a*(sin(t)-t*cos(t));
> a:=1;
> specify the value of
> a for plotting
> plot([x(t),y(t),t=0..2*Pi]);
> a:='a';
> change
> a back into a general parameter

Take a look at the plot command. Inside the square brackets we have first the
expression x(t) describing the x coordinate, then y(t) describing the y coordinate,
and finally the parameter interval. Maple will show a region of the xy-plane just
large enough to contain the curve. If we wish we can specify x and y coordinates
intervals for the plot (after the square brackets). Now, we compute the arc length.

> l:=Int(sqrt((D(x)(t))^2+(D(y)(t))^2), t=0..2*Pi);
> value(%);

If a curve is represented by an equation using the polar coordinates, for example r =
f(θ) for θ in [α, β], then the arc length formula becomes l(γ) = ∫_α^β \sqrt{D(f)(θ)² + f(θ)²} dθ.

**Example 4.** Find the length of the cardioid given by the polar equation r(θ) =
a(1+cos(θ)) for positive real numbers a and θ in [0, 2π].

Maple can plot curves in polar coordinates. The form is similar to the ordinary
(Cartesian) parametric plot command, with the coords=polar option at the end.

> r:=theta->a*(1+cos(theta));
> a:=5;
> plot([r(t),t,t=0..2*Pi],coords=polar);
> a:='a';

This plots the curve r = a(1 + cos(θ)) for θ in [0, 2π], which is really a special case
of a polar parametric plot. Inside the list which forms the first imput for plot we
have first the definition of r as a function of t, then θ as a function of t, and then the
parameter interval. In our case, we have θ = t, but in general both r and θ could
depend on the parameter t in an arbitrary way. After the square bracket we can also
specify x and y intervals for the plot, constrained scaling (which is usually a good
idea for polar plots), or other options.

The arc length can be computed directly from the formula derived in the text book.

> l:=Int(sqrt((D(r)(theta))^2+(r(theta))^2), theta=0..2*Pi);
> value(%) ;

2. Areas Bounded by Curves

With Maple’s help in plotting parametric and polar curves, and with the area formulas
derived in the textbook, it is now not hard to compute the areas bounded by curves.
This section consists of several examples.
Example 5. Compute the area of the region lying in the first quadrant inside the circle \( x^2 + y^2 = 3a^2 \) and bounded by the parabolas \( x^2 = 2ay \) and \( y^2 = 2ax \), where \( a \) is positive.

\[
\begin{align*}
&> \text{eq1:=} x^2+y^2=3*a^2; \\
&> \text{eq2:=} x^2=2*a*y; \\
&> \text{eq3:=} y^2=2*a*x; \\
&> a:=5; \\
&> \text{implicitplot(\{eq1,eq2,eq3\},x=0..10, y=0..10);} \\
&> a:=’a’;
\end{align*}
\]

Which curve is for which function? We concentrate on the first quadrant. Then we can write the three curves as:

\[
f_1(x) = \sqrt{3a^2-x^2}, \quad f_2(x) = \frac{x^2}{2a}, \quad f_3(x) = \sqrt{2ax}.
\]

Solve the three equations \( f_2(x) = f_3(x) \), \( f_1(x) = f_3(x) \) and \( f_1(x) = f_2(x) \) for \( x \) to get the intersection points \( O=[0,0] \), \( A=[a, \sqrt{2}a] \), \( B=[\sqrt{2}a, a] \) and \( C=[2a, 2a] \). Note that we only consider the first quadrant.

The required area is enclosed by OAB, which can be expressed as

\[
A = \int_0^a f_3(x) - f_2(x) \, dx + \int_a^{\sqrt{3}a} f_1(x) - f_2(x) \, dx.
\]

When the area of the region is bounded by a closed parametric curve \( r(t) = (x(t), y(t)) \), with \( r(\alpha) = r(\beta) \), the area can be computed by

\[
A = -\int_\alpha^\beta y(t) \frac{D(x)}{D(y)}(t) \, dt = \int_\alpha^\beta x(t) \frac{D(y)}{D(x)}(t) \, dt = \int_\alpha^\beta x(t) \, D(y)(t) - y(t) \, D(x)(t) \, dt.
\]

Example 6. Compute the area enclosed by the cardioid given by the closed parametric curve: \( a \cos(t) \left(1 + \cos(t)\right) \) and \( a \sin(t) \left(1 + \cos(t)\right) \) for \( t \) in \([-\pi, \pi]\).

\[
\begin{align*}
&> x:=t->a*cos(t)*(1+cos(t)); \\
&> y:=t->a*sin(t)*(1+cos(t)); \\
&> a:=5; \\
&> \text{plot([x(t),y(t),t=-Pi..Pi]);} \\
&> a:=’a’; \\
&> A:=1/2*Int((x(t)*D(y)(t)-y(t)*D(x)(t)), t=-Pi..Pi);
\end{align*}
\]
Finally, we give an example of the region bounded by a polar curve.

**Example 7.** Find the area of the figure cut out by the circle $r_1(\theta) = \sqrt{3} \sin(\theta)$ from the cardioid $r_2(\theta) = 1 + \cos(\theta)$.

```maple
> r1:=theta->sqrt(3)*sin(theta);
> r2:=theta->1+cos(theta);
> p1:=plot([r1(theta),theta,theta=0..2*Pi], coords=polar):
> p2:=plot([r2(theta),theta,theta=0..2*Pi], coords=polar):
> plots[display]({p1,p2});
```

Solve the equation $r_1(\theta) = r_2(\theta)$ for the intersection points.

```maple
> solve(r1(theta)=r2(theta),theta);
```

The answers are $\theta = \pi$ and $\theta = \frac{2\pi}{3}$. Thus the intersection points are $O=[0,0]$ and $A = \left[ \frac{3}{4}, \frac{3\sqrt{3}}{4} \right]$, since $\sin(\pi) = 0$, $\cos(\pi) = -1$, $\sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{2\pi}{3}) = \frac{1}{2}$, and $x = r \cos(\theta)$, $y = r \sin(\theta)$. The region in question consists of two sectors: one is the circular sector, the other a sector of the cardioid. Both sectors adjoin each other along the ray $\theta = \frac{2\pi}{3}$.

```maple
> A:=1/2*Int(r1(theta)^2,theta=0..Pi/3)+1/2*Int(r2(theta)^2,theta=Pi/3..Pi);
> value(%);
```

### 3. Exercises

Compute the length of the following arcs (questions 1 to 4):

- **1.** The curve $y = \frac{x^2}{2} - 1$ cut off by the $x$-axis.

- **2.** The curve $y = \frac{x\sqrt{x^2-1} - \ln(x + \sqrt{x^2-1})}{2}$ between $x = 1$ and $x = a + 1$.

- **3.** The parametric curve $x = \frac{t^6}{6}$, $y = 2 - \frac{t^4}{4}$ between the points of intersection with the axes of coordinates.

- **4.** The closed polar curve $r = a \sin(\theta)^{3}$ where $a$ is positive.

Compute the area of the following regions (questions 5 to 10):

- **5.** The figure lying in the first quadrant and bounded by the curves $y^2 = 4x$, $x^2 = 4y$ and $x^2 + y^2 = 5$. 

• 6. The figure bounded by the parabola \( y = -x^2 - 2x + 3 \), the line tangent to it at the point (2,-5) and the y-axis.

• 7. The region enclosed by the parametric curve \( x = a \sin(t), y = b \sin(2t) \).

• 8. The region enclosed by the loop of the curve \( x = t^2, y = t - \frac{t^2}{3} \).

• 9. The figure bounded by the cardioid \( r = a(1 - \cos(\theta)) \) and the circle \( r = a \) for positive real number \( a \).

• 10. The region enclosed by the curve \( r = a \sin(\theta) \cos(\theta)^2 \), where \( a \) is a positive real number.

Lab 32. Applications of Definite Integral (II)
Continuing from the previous lab, this labs moves on to the computation of volumes and surface areas of revolution. Some ability to imagine three dimensional graphs is required in this lab. We may also try to use Maple to plot the 3D graphs for the volumes and the surface areas. However, we won’t discuss the 3D-plotting technique in this lab.

1. Computations of Volumes
Basically, we use the elementary formula \( V = S(R)h \) for the volume of a cylinder, which changes to \( V = \int_a^b S(x) \, dx \) for a general solid. According to the situation at hand, we have different forms of the general formula.

If we revolve the plane region \( R \) bounded by \( y = f(x), y = 0, x = a \) and \( x = b \) about the \( x \)-axis, then the volume of the revolution solid \( D \) is \( V(D) = \pi \int_a^b f(x)^2 \, dx \). When \( R \) is bounded by curves \( y = f(x), y = g(x), x = a \) and \( x = b \), where \( f(x) \) and \( g(x) \) are non-negative, with \( g(x) \leq f(x) \), the volume of the solid \( D \) obtained by rotating the region \( R \) about the \( x \)-axis is \( V(D) = \pi \int_a^b f(x)^2 - g(x)^2 \, dx \). If the region \( R \) is bounded by a closed, smooth and positively oriented parametric curve given by \( r(t) = (x(t), y(t)) \) with \( r(\alpha) = r(\beta) \), then the volume of the revolution solid about the \( x \)-axis: \( V(D) = -\pi \int_{\alpha}^{\beta} y(t)^2 \, D(x)(t) \, dt \), the volume of the revolution about the \( y \)-axis: \( V(D) = -\pi \int_{\alpha}^{\beta} x(t)^2 \, D(y)(t) \, dt \). If the region \( R \) is enclosed by a closed smooth polar curve \( r = f(\theta) \) for \( \theta \) in \([\alpha, \beta]\), the solid \( D \) is obtained by revolving \( R \)
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about the line \( y = \rho, \rho > \max\{ f(\theta)|\theta \in [\alpha, \beta]\} \), then \( V(D) = -\pi \int_\alpha^\beta (f(\theta)\sin(\theta) + \rho)^2 \left(D(f)(\theta)\cos(\theta) - f(\theta)\sin(\theta)\right) d\theta. \)

**Example 1.** Find the volume of the solid obtained by revolving the region under the catenary curve \( y = a \cosh\left(\frac{x}{a}\right) \) between 0 and \( x \) about the \( x \)-axis.

\[
> f := x -> a*cosh(x/a);
> a := 2;
> plot(f(x),x=0..10);
> a := 'a';
\]

Imagine the solid by revolving the region \( R \) about the \( x \)-axis.

\[
> V := Pi*Int(f(t)^2,t=0..x);
> value(%) ;
> expand(%) ;
> simplify(%,exp);
\]

**Example 2.** Consider the region \( R \) bounded by the cycloid \( x = a(t - \sin(t)), \ y = a(1 - \cos(t)) \) and the \( x \)-axis. Find the volume of the solid \( D \) obtained by revolving \( R \) about the \( x \)-axis.

\[
> x := t -> a*(t-sin(t));
> y := t -> a*(1-cos(t));
> a := 5;
> plot([x(t),y(t),t=0..2*Pi]);
> a := 'a';
\]

Compute the volume by applying the formula \( V = -\pi \int_\alpha^\beta y(t)^2 D(x)(t) \, dt. \)

\[
> V := Pi*Int(y(t)^2*D(x)(t),t=0..2*Pi);
> value(%) ;
\]

**Example 3.** The region \( R \) is enclosed by the polar curve \( r = a \cos(\theta)^2 \). Find the volume of the solid \( D \) obtained by revolving \( R \) about the \( x \)-axis.

\[
> r := theta -> a*cos(theta)^2;
> a := 5;
> plot([r(theta),theta,theta=-Pi..Pi], coords=polar);
> a := 'a';
\]

Notice that there are two closed regions, which are symmetric about the \( y \)-axis. The curve is written in polar coordinates. Since the solid is the revolution about the \( x \)-axis, it would be better to write the curve in Cartesian coordinates, so that we can apply the formula \( V(D) = \pi \int_a^b f(x)^2 \, dx \), i.e. \( V(D) = \int_a^b y^2 \, dx \). Maple doesn’t have built-in commands to transform from polar to Cartesian coordinates, but it is not hard to do, using the formulas \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \). We can write the above polar curve in the following parametric form: \( x = a \cos(\theta)^3 \) and \( y = a \cos(\theta)^2 \sin(\theta) \) for \( \theta \) in \([\pi, \pi]\), while \( x \) changing from \( x = -a \) to \( x = a \). Check the graph to see if our parametric curve is the same as the original polar curve.

\[
> x := t -> a*cos(t)^3;
\]
> y:=t->a*cos(t)^2*sin(t);
> a:=5;
> plot([x(t),y(t),t=-Pi..Pi]);
> a:='a';

Now, we can apply the formula \( V = \pi \int_0^a y^2 \, dx \). Furthermore, we know that \( dx(t) = D(x(t)) \, dt \). When \( x \) changes from \( x = 0 \) to \( x = a \), the corresponding parameter \( \theta \) changes from \( \theta = \frac{\pi}{2} \) to \( \theta = 0 \).

> V:=2*Pi*Int(y(t)^2*D(x)(t),t=Pi/2..0);
> value(%);

**Example 4.** The region \( R \) is enclosed by \( y = x \) and \( y = x^2 \) from \( y = 0 \) to \( y = 1 \). Find the volume of the solid obtained by revolving the region \( R \) about the \( y \)-axis.

> f1:=x->x;
> f2:=x->x^2;
> plot({f1(x),f2(x)},x=0..1);

There are two ways to understand the construction of the solid, which deduce two different formulas to compute the volume.

> V1:=2*Pi*(Int(x*f1(x),x=0..1)-Int(x*f2(x),x=0..1));
> value(%);
> g1:=y->y;
> g2:=y->sqrt(y);
> v2:=Pi*(Int(g2(y)^2-g1(y)^2,y=0..1));
> value(%);

The answers for \( V1 \) and \( V2 \) should be the same. Try to understand why they are both right.

2. Area of Surface of Revolution

Suppose that \( r(t) = (x(t), y(t)) \) for \( t \) in \([a, b] \) is a smooth parametric curve. Revolve this curve about the \( x \)-axis. Then the resulting surface of this revolution is \( A = 2 \pi \int_a^b y(t) \sqrt{D(x(t))^2 + D(y(t))^2} \, dt \).

**Example 5.** Compute the area of the surface formed by revolving about the cycloid \( x = a(t - \sin(t)) \) and \( y = a(1 - \cos(t)) \) for \( t \) in \([0, 2\pi] \) about the \( x \)-axis.

Apply the above formula directly.

> x:=t->a*(t-sin(t));
> y:=t->a*(1-cos(t));
> A:=2*Pi*Int(y(t)*sqrt(D(x)(t)^2+D(y)(t)^2), t=0..2*Pi);
> value(%);

**Example 6.** Compute the surface area of the ellipsoid defined by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \), for \( a > b > 0 \).
This ellipsoid is generated by revolving the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) about the x-axis. We can write the ellipse equation as the following parametric equations \( x = a \cos(t) \) and \( y = b \sin(t) \) for \( t \) in \([0, \pi]\). The command \texttt{value(\%)} doesn’t help for this integral. We need to do the computation by ourselves.

\[
A := \text{simplify}(A, \text{trig});
\]
\[
\text{with(student)}:
\]
\[
\text{changevar(cos(t)=x,A,x)};
\]
\[
\text{value(\%)};
\]

3. Exercises

Compute the volumes of the following solids (questions 1 to 4)

- 1. The solid spherical segment of two bases cut out by the planes \( x = 2 \) and \( x = 3 \) from the sphere \( x^2 + y^2 + z^2 = 16 \).

- 2. The solid of revolution generated by revolving a figure about the y-axis, while the figure is bounded by an arc of the sinusoid \( y = \sin(x) \), the x-axis and the line \( y = 1 \).

- 3. The solid obtained by revolving the cardioid \( r = a (1 + \cos(\theta)) \) for positive \( a \) about the x-axis.

- 4. The solid generated by revolving the region enclosed by the loop of the curve \( x = at^2 \) and \( y = a(t - \frac{t^3}{3}) \) about the x-axis.

Compute the areas of the following surfaces (questions 5 to 7)

- 5. The surface obtained by revolving the ellipse \( 4x^2 + y^2 = 4 \) about the y-axis.

- 6. The surface generated by revolving an arc of the curve \( x = t^2 \) and \( y = \frac{t(t^2-3)}{3} \) between the points of intersection of the curve and the x-axis about the x-axis.
• 7. The surface generated by revolving the polar curve \( r = 2a \sin(\theta) \) for positive number \( a \), about the \( x \)-axis.

For the following questions (questions 8 to 11), using Maple plot of the specified region \( R \), revolve \( R \) about the indicated axis to create a solid of revolution and sketch this solid by hand. Set up the formula for the volume of this solid and compute the volume.

• 8. \( f(x) = \sqrt{x^2 + 1} \) for \( x \) in [1, 2]. \( R \) is the region between the \( x \)-axis and the graph of \( f(x) \). Revolve \( R \) about the \( y \)-axis.

• 9. \( f(x) = \sqrt{x^2 + 1} \) for \( x \) in [1, 2]. \( R \) is the region between this graph and the \( y \)-axis. Revolve \( R \) about the \( y \)-axis.

• 10. \( f(x) = x \sin(x) \) for \( x \) in [0, \( \pi \)]. \( R \) is the region between the \( x \)-axis and the graph of \( f(x) \). Revolve \( R \) about the \( x \)-axis.

• 11. \( R \) is the region between the curves \( f(x) = \cos(x) \) and \( g(x) = \sin(x) \) between two successive intersections. Rotate \( R \) about the line \( y = -2 \).

**Lab 33. Infinite Series**

In this Lab, we will learn how to use Maple to determine whether an infinite series converges or not. There are a variety of tests that indicate whether a particular summation is convergent or divergent. Some of these are investigated in this Lab.

1. Series in Maple

Maple has its own facilities for handling series. In practice, we would rely on these most of the time. However, in most of this lab we want to learn how to do the analysis ourselves, using Maple as an aid in the calculations.

Maple’s sum function can be used for both partial sums and infinite series.

\[
> \text{sum}(1/2^k, k=1..6);
\]

\[
> \text{sum}(1/2^n, n=1..k);
\]

The sum command has an inert form \text{Sum} which returns the sum unevaluated.

\[
> \text{Sum}(1/2^n, n=1..k);
\]

\[
> \text{value}(\%);
\]
> Limit(%,k=infinity);
> value(%);

The above four commands result to the same answer as the following one command does.
> sum(1/2^k,k=1..infinity);

Maple can find the sums of some infinite series in closed form, i.e. as an exact expression. If it can’t do this, sum just return the series. Even if there is no exact expression for the sum, evalf may be able to approximate the sum.
> sum(k^2/(k^4+k+1),k=1..infinity);
> evalf(%);

However, sometimes evalf doesn’t come up with an answer either.
> evalf(Sum(k^3/(k^4+k+1),k=1..infinity));
> evalf(Sum(ln(k+1)/k^3,k=1..infinity));

As we know, graph is always very helpful. To investigate the behavior of some partial sums as a function of $k$, we can plot some sample points. The following little procedure computes the partial sums and can be used to create a list of sample points $[i, S(i)]$.
> S:=proc(k) local n: Sum(1/n,n=1..k)
> end:

Now, we use the procedure to create 40 sample points and plot them out.
> pts:=[seq([10*i+1,S(10*i+1)],i=1..40)]:
> plot(value(pts),style=point);

From the graph, we see that the partial sums do not appear to approach a finite limit as $i \to \infty$, and in fact they never do. We can plot the sums $\sum_{n=1}^{k} \frac{1}{n}$ for large $k$'s.
> pts:=[seq([100*i+1,S(100*i+1)],i=1..40)]:
> plot(value(pts),style=point);

From this, we can guess that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. However, to determine whether an infinite series converges, we need a more theoretical proof, which is the task of the next section.

2. Convergence Tests

Maple does not have a specific command to determine whether a series converges or diverges. However, it can be used to apply the standard convergence tests for a series, such as the integral test, comparison test, ratio test and root test.

Example 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

The function $f(x) = \frac{1}{x}$ is continuous, positive, and decreasing, we can examine the corresponding integral instead of the partial sums.
> f:=n->1/n;
> int(f(x),x=1..infinity);

The answer is infinite, so by the integral test, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Example 2. Does $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$ converges?
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Again, the function \( f(x) = \frac{1}{x^2+1} \) is continuous, positive, and decreasing, so we can try the integral test.

\[
> f:=n->1/(n^2+1);
> int(f(x),x=1..infinity);
\]

It is finite, so by the integral test, our series converges.

Integration of a complicated function is often time-consuming and not always successful, so the integral test is probably not the best choice. We can try the comparison test. The usual choice for comparison is a \( p \)-series \( \sum k^{(-p)} \). For a rational function, it is easy to see which \( p \) to use.

**Example 3.** Does \( \sum_{k=1}^{\infty} \frac{k^3}{k^4+k+1} \) converge?

In the expression of the term \( f(k) = \frac{k^3}{k^4+k+1} \), the numerator has degree 3 and the denominator has degree of 4. Thus, we choose \( g(k) = \frac{k^3}{k^4} = \frac{1}{k} \) for the comparison test.

We would want to show that \( Ag(k) \leq f(k) \) for all sufficiently large \( k \), where \( A \) is a positive constant. After some experiments, we figure out that \( A = \frac{1}{2} \) is good enough.

\[
> f:=k->k^3/(k^4+k+1);
> f(k)-1/(2*k);
> normal(%);
\]

For \( 2 \leq k, 0 \leq k^4 - k - 1 \), therefore \( \frac{1}{2k} \leq f(k) \) for all large \( k \). Since \( \sum \frac{1}{k} \) diverges, so does our series.

We can also try the limit comparison test.

\[
> \text{limit}(f(k)/k^{-1},k=infinity);
\]

Since the limit exists and is not zero, and \( \sum \frac{1}{k} \) diverges, our series diverges as well.

**Example 4.** Does \( \sum_{k=1}^{\infty} \frac{k \ln(k^2+1)}{k^{3/2}} \) converge?

The denominator of the term has degree of 3. What is the degree of the numerator?

We know that for large \( x, 1 < \ln(x) < x \). Thus the degree of the numerator is something between 1 and 2. We can try \( \frac{3}{2} \) for the numerator, which leads us to choose \( g(k) = \frac{k^{(3/2)}}{k^3} = \frac{1}{k^{(3/2)}} \) for the comparison test.

\[
> f:=k->k*ln(k^2+1)/(k^3-k+1);
> f(k)-1/(2*k);
> \text{normal}(%);
> \text{limit}(f(k)/k^{(-3/2)},k=infinity);
\]

The limit is finite and \( \sum \frac{k^{(-3/2)}}{k} \) converges, so our series converges as well.

Series involving \( k \)'th powers or factorials are usually good candidates for the ratio test or the root test.

**Example 5.** For what values of the constant \( a \) does \( \sum_{k=1}^{\infty} \frac{a^k (k+1)(2k)}{(2k+1)!} \) converge?

We'll try the ratio test here (the root test would give the same answer). Let \( b = |a| \) (the test is actually for absolute convergence).

\[
> f:=k->b^k*(k+1)*(2k)/(2k+1)!
> \text{limit}(f(k+1)/f(k),k=infinity);
\]

The series converges absolutely when \( |a| < \frac{4}{e^2} \) and diverges when \( \frac{4}{e^2} < |a| \). What about \( |a| = \frac{4}{e^2} \)?
We want to try a comparison test with a $p$-series for the case $|a| = \frac{4}{e^2}$, but it is not clear which $p$ to use. Observe that if $f(k)$ was $ck^{(-p)}$, we would have $\ln(f(k)) = \ln(c) - p\ln(k)$, so $\frac{\ln(f(k))}{\ln(k)} \rightarrow -p$ as $k \rightarrow \infty$.

```plaintext
> b:=4/exp(2);
> assume(k,integer);
> expand(ln(f(k))/ln(k));
> normal(%);
> limit(%,k=infinity);
```

From the answer of the above limit, it seems most likely that $f(k)$ behaves like $k^{-\frac{3}{2}}$ as $k \rightarrow \infty$. We can confirm it by taking the following limit.

```plaintext
> expand(f(k)*k^(3/2));
> limit(%,k=infinity);
```

Since the limit is finite, the series converges absolutely for $|a| = \frac{4}{e^2}$.

**Example 6.** Test for convergence of the series $\sum_{n=1}^{\infty} \left(\frac{7n+2}{9n+2}\right)^n$.

```plaintext
> f:=n->((7*n+3)/(9*n+2))^n;
> limit(f(n)^(1/n),n=infinity);
```

The limit of the $n$th root is finite, so this series converges absolutely.

### 3. Exercises
Test for convergence of the following series (questions 1 to 14) using the integral test, comparison test, ratio test or root test.

- 1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$; 2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for real number $p$;
- 3. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2^n}$; 4. $\sum_{n=1}^{\infty} \frac{3n+13}{(11n-2)^n}$;
- 5. $\sum_{n=1}^{\infty} \frac{\ln(n)^2}{9n^2}$; 6. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$;
- 7. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$; 8. $\sum_{n=1}^{\infty} \frac{5}{3n^2+5n+4}$;
- 9. $\sum_{n=1}^{\infty} \frac{5}{3n^2}$; 10. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$;
- 11. $\sum_{n=1}^{\infty} \frac{2n^3+5n}{\sqrt{n+3n^2}}$; 12. $\sum_{n=1}^{\infty} e^{(-n)} n!$;
- 13. $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^4}$; 14. $\sum_{n=1}^{\infty} \frac{(n+2)!}{n! 10^n}$. 

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15. Find the partial sums $s_n$ of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ for $n = 10, 20, 40$ and $80$. Examine the differences between these. What trends do you notice? Assuming these trends continue, what can you say about the convergence of the series and about its sum?

16. Test the following series for convergence.
   
   a). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2-1}}$; b). $\sum_{n=1}^{\infty} \frac{n}{2^n-n}$;
   
   c). $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$; d). $\sum_{n=1}^{\infty} \frac{(-1)^n}{m(n+1)}$.

17. a). Suppose $a_k > 0$ for all $k$ and $\lim_{k \to \infty} \frac{\ln(ka_k)}{\ln(m(k))} = p$ (possibly $\infty$ or $-\infty$). Show that $\sum_{k=1}^{\infty} a_k$ diverges if $p > -1$ and converges if $p < -1$.
   
   b). Use this loglog test to determine whether $\sum_{k=1}^{\infty} \frac{k^{(2k)}(k^2-1)!}{(k^2+k)!}$ converges.
   
   c). Do the same for $\sum_{k=1}^{\infty} \frac{k^{(2k)}(2k^2)!}{2^{(k^2)}(k^2+k)!}$.
   
   d). Construct, using powers, logarithms and/or factorials, a series $\sum a_k$ with $a_k > 0$ for which this test is inconclusive.

18. For what values of the constant $p$ does $\sum_{n=1}^{\infty} \frac{(2n)!n^p}{4^n n!}$ converge? 
   
   Hint: you can use \texttt{asympt} on $\frac{a_n}{n^p}$ if it doesn’t work on $a_n$ directly.

19. a). Does the series $\sum_{n=2}^{\infty} \frac{1}{(-1)^n \sqrt{n+1}}$ converge? 
   
   Hint: this is not an alternating series. Group the terms into pairs.
   
   b). Now try it for $\sum_{n=2}^{\infty} \frac{1}{(-1)^n n+1}$.

20. Let $a_0 = 2$ and $a_{n+1} = 2 - \sqrt{4-a_n}$ for $0 \leq n$. Show that $\sum_{n=0}^{\infty} a_n$ converges. 
   
   Hint: first show that $a_n$ is decreasing. The ratio test works well if you express $a_n$ in terms of $a_{n+1}$ rather than the reverse.
Lab 34. Powers Series

In this Lab, we introduce the power series. We will learn how to manipulate and use power series. Taylor series are a special and very useful power series. For the rest of this Lab, we give some examples on how to obtain, use and manipulate Taylor series in Maple. We will also use some graphical methods to examine the convergence of Taylor series.

1. Power Series

A power series in \( x \) is a series of the form \( \sum_{n=0}^{\infty} c_n x^n \). Each value for \( x \) results in a series. This may or may not converge, depending on the value of \( x \) and the sequence of coefficients \([c_n]\). Power series can be used to define functions of \( x \).

\[
> f:=x->\text{sum}(a[i]*x^i,i=0..\infty);
\]

For example, \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) is valid for those values of \( x \) for which the power series converges. We can identify the \( x \) values for which this converges by using the ratio test.

\[
> f:=n->x^n;
\]

\[
> f(n+1)/f(n);
\]

\[
> \text{abs(simplify(%))};
\]

The answer \(|x|\) tells us that when \(|x| < 1\) the series converges and when \(|x| > 1\) the series diverges. It has a radius of convergence of 1.

2. Constructing Power Series

There are two basic approaches to construct power series for some functions. One is to construct a power series by modifying an existing series, or by combining two or more known series. The other is to construct a formula for the \( n \)th term of the series.

Example 1. Find a power series representation for \( \sin(x) \cos(2x) \).

We first find the series representations for \( \sin(x) \) and \( \cos(2x) \) using \texttt{series} command.

\[
> \text{series}(\sin(x),x,8);
\]

\[
> \text{series}(\cos(2*x),x,8);
\]

Then multiply them together.

\[
> %*%;
\]

\[
> \text{like multiplying two polynomials}
\]

\[
> \text{series}(%,x,8);
\]

\[
> \text{extract the terms using \texttt{series} command}
\]

Example 2. Use polynomial arithmetic to show that the power series for \( \frac{\sin(x)}{\cos(x)} \) begins as \( x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7) \).

We wish to have values for \( a_0, a_1, \ldots \) so that \( \frac{\sin(x)}{\cos(x)} = (\sum_{i=0}^{6} a_i x^i) + O(x^7) \).

\[
> \text{sin(x)}/\text{cos(x)}=\text{sum}(a[i]*x^i,i=0..6)+0(x^7);
\]

\[
> %*\text{cos(x)};
\]

For power series, the above equation implies:
> series(sin(x),x)=series(cos(x),x)* (sum(a[i]*x^i,i=0..6)+O(x^7));
> series(lhs(%),x)=series(rhs(%),x);
> series(lhs(%)-rhs(%),x);
> convert(%,polynom);
> coeffs(%,x);

The coefficients of the various powers of \( x \) must be 0.

> solve(%);

With the answer \( a_0 = 0, a_1 = 1, a_2 = 0, a_3 = \frac{1}{3}, a_4 = 0 \) and \( a_5 = \frac{2}{15} \), we have proved that

\[
\frac{\sin(x)}{\cos(x)} = x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7).
\]

3. Taylor Series

Maple has the \texttt{taylor} command for finding the Taylor series of an expression.

> taylor(exp(x),x=c,5);

If we need the Taylor polynomial, we must use the command \texttt{convert}.

> convert(%,polynom);

Not all expressions have Taylor series. Maple will return an error message if we ask it for a Taylor series that doesn’t exist.

> taylor(sqrt(x),x=0);

As we know from the previous section, various operations can be done to obtain new series from old series: the basic operations of arithmetic, as well as substitution, differentiation and integration. Most of the operations are actually done with the Taylor polynomials rather than Maple’s series structures.

**Example 3.** Obtain the degree 10 Taylor polynomial for \( \ln(1 + x^2) \cos(\cos(x)) \) in powers of \( x \), starting from the well known Taylor polynomials for \( \frac{1}{1+t} \) and \( \cos(t) \).

We begin with \( \ln(1 + x^2) \), which can be obtained from \( \ln(1 + t) \) by substituting \( t = x^2 \).

We’ll derive the series for \( \ln(1 + t) \) from the series for its derivative \( \frac{1}{1+t} \).

> taylor(1/(1+t),t=0);
> int(%,t);

Before we substitute \( t = x^2 \) into the series, we must convert it to a polynomial.

> s1:=subs(t=x^2,convert(%,polynom));

Note that the remainder will be \( O(x^{14}) \).

Now, for the series for \( \cos(\cos(x)) \).

> s2:=taylor(cos(x),x=0,11);

\( \cos(x) \) is not near 0 when \( x \) is near 0, so \( O(\cos(x)^n) \) would not be small. Thus, we need to use the series for \( \cos(t) \) in powers of \( t - 1 \), because \( \cos(0) = 1 \).

> taylor(cos(t),t=1);

Substitute \( t = s2 \) in this series (after converting both to polynomials).

> subs(t=convert(s2,polynom),convert(%, polynom));

Then we use \texttt{taylor} to extract the terms we’re interested in.
After converting this to a polynomial, we multiply it by \( s1 \), which was the Taylor polynomial for \( \ln(1 + x^2) \). Then we extract the degree 10 Taylor polynomial of the product.

\[
\text{convert(\%\%,\text{polynom})*s1,x=0,11);}
\]

\[
\text{convert(\%\%,\text{polynom});}
\]

\[
\text{evalf(\%);}\]

4. Convergence of Maclaurin Series
Previously we have studied the convergence of a series of numbers to its limit, here we have a series of functions converging to a function. In this section, we explore graphically the way the Maclaurin series of a function converges to the function. First we set up some functions for conveniently finding Maclaurin polynomials and evaluating them at a point.

\[
P(\text{n},t) -> \text{subs(x=t,convert(taylor(f(x),x=0, n+1),\text{polynom}))};
\]

\[
R(\text{n},t) -> f(t)-P(\text{n},t);
\]

\( P(n,t) \) will be the Maclaurin polynomial \( P_n(t) \), and \( R(n,t) \) will be the error \( R_n(t) \).

**Example 4.** Study the convergence of the Maclaurin series for the exponential function \( \text{exp} \).

We plot \( e^x \) and its Maclaurin polynomials \( P_n(x) \) of degrees 1 through 10.

\[
f:=\text{exp} ;
\]

\[
\text{plot} \{\text{f(x)},\text{seq}(P(\text{nn},x),\text{nn}=1..10)\},x=-6..2, y=-2..8);
\]

Each Maclaurin polynomial is quite close to \( f(x) \) on some interval around 0, but eventually moves away from \( f(x) \) as \( x \) gets away from 0. For any given \( x \), \( P_n(x) \) goes to \( f(x) \) as \( n \to \infty \). The larger \( n \) is, the larger the interval on which \( P_n(x) \) is close to \( f(x) \).

Next, we produce an animation of the Maclaurin polynomials approaching \( f(x) \). This time we don’t use the **animate** command for Maple.

\[
\text{display}([\text{seq(display([p1,plot(P(\text{nn},x),x=-6..2,y=-2..8,colour=red) ])},\text{nn}=1..12])}, \text{insequence=true});
\]

After you click on the graph and click the play button, you can see the approaching of the Maclaurin polynomials to the function \( f(x) \).

Now, we plot the remainders.

\[
\text{plot} \{\text{seq}(R(\text{nn},x),\text{nn}=1..10)\},x=-6..6, y=-3..3);
\]

We might take a closer look, using an animation.

\[
\text{display}([\text{seq(plot(R(\text{nn},x),x=-6..6,y=-3..3) },\text{nn}=1..10])}, \text{insequence=true});
\]

Try to understand why it behaves this way.
5. Exercises

- 1. For each of the following functions $f(x)$, plot the function and several of its Maclaurin polynomials $P_n(x)$ on the same graph, using appropriate intervals, to see how $P_n(x)$ converges to $f(x)$. Then plot the remainders $R_n(x)$. As $n$ increases, what happens to the interval around 0 on which $|R_n(x)| < .1$? Hint: if the function is even or odd, you need only consider even or odd partial sums, respectively.
  a). $\sin(x)$
  b). $\tan(x)$
  c). $\ln(1 + x)$
  d). $\sin(x^2)$
  e). $\frac{x}{1+x^2}$

- 2. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$.
  a). What, according to Maple, is the sum $f$ of this series? What is the radius of convergence?
  b). Plot the remainders $R_n(x)$ for several different $n$. For what $x$ values do they appear to be converging to 0?
  c). How many terms of the series would be needed to approximate the sum to an accuracy of 0.001 for $x = .5$? For $x = .9$?

- 3. Starting with ”well-known” series, obtain the degree 9 Maclaurin polynomial for $f(x) = \cos(\sin(x)\ln(1 + x))$.

- 4. a). Let $f(x) = \frac{\int_{0}^{2\pi} e^{(x \cos(t))} dt}{2\pi}$. Maple can’t evaluate this integral, but in fact it turns out to be a Maple function: BesselI(0,x). Evaluate $f$ and BesselI(0,...) at several points to convince yourself that they are the same.
  b). Find the Maclaurin series of $f(x)$ by starting with the Maclaurin series of exp. Hint: $\int_{0}^{2\pi} \cos(t)^n dt = 0$ if $n$ is odd and $\frac{2^{1-n} \pi n!}{((n/2)^n)}$ if $n$ is even.
  c). Check your answer for b) by using taylor on $f(x)$ and BesselI(0,x).
  d). Plot BesselI(0,x) and its Maclaurin polynomials $P_n(x)$ for even $n$ from 2 to 10 and $x$ from 0 to 5.
  e). The function $f(x)$ satisties the differential equation $x (D^{(2)})(f)(x)+D(f)(x)−x f(x) = 0$. What is $x (D^{(2)})(P_{10}(x)) + D(P_{10}(x)) − x P_{10}(x)$?
5. The first few terms of two particular power series are given as \( f(x) = 1 + x + x^2 + 2x^3 + 3x^4 + O(x^5) \), and \( g(x) = 3 + 5x + 7x^2 + 9x^3 + 11x^4 + O(x^5) \). Find as many terms as possible of the power series corresponding to the product \( f(x)g(x) \).

6. Find values of \( a \), \( b \), and \( c \) so that the first few terms of the series of \( \frac{a + 3x}{1 + bx + c x^2} \) to 4 degree are given by \( 1 + 5x + 5x^2 - 15x^3 - 55x^4 + O(x^5) \).

7. Find a power series representation for \( \sin(5x^2 + x) \).

8. Obtain the series representation to order 20 of \( \sin(x^2) \) expanded about \( x = 0 \). Use this series representation to obtain a series representation of \( \int \sin(x^2) \, dx \). Note that this particular antiderivative cannot be represented in terms of polynomials, logarithms, and exponentials and the standard trigonometric functions we have studied in this course.