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On the exact distribution of the sum of the largest $n - k$ out of n normal random variables with differing mean values

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Motivated by an application in Electrical Engineering, we derive the exact distribution of the sum of the largest $n - k$ out of n normally distributed random variables, with differing mean values. Comparisons are made with two normal approximations to this distribution – one arising from the asymptotic negligibility of the omitted order statistics and one from the theory of L-statistics. The latter approximation is found to be in excellent agreement with the exact distribution.

Keywords: L-statistics; Order statistics

MSC 2000: Primary: 62E15, 62G30; Secondary: 62E20

1. Introduction

A problem of some interest in Electrical Engineering is the following. Future cellular phone receivers will select the strongest $n - k$ out of n received signals from n antennae and process these signals. When n independent antennae are used, this antenna diversity system greatly reduces the effects of signal fading, particularly when the signals have differing mean values. The performance analysis of this system requires determining the distribution of the sum of the strongest $n - k$ out of n processed signals, which are normally distributed.

The dominant noise corrupting a signal in a radio receiver is the noise generated by the receiver itself. The oscillator and front-end amplifiers in the radio receiver generate noise that is amplified by subsequent stages of amplifiers. Thus, the noise power (variance σ^2) is the same for all signals as it comes from the receiver. The signal mean values μ_i differ because the transmitted signal strength depends upon the transmission path, distance, and so on.

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Formally, if $X_1, \dots, X_n$ are independent random variables, with $X_i \sim N(\mu_i, \sigma^2)$, and if $X_{1:n} < \cdots < X_{n:n}$ are the order statistics, then we seek the distribution of

$$Y_{n-k} := \sum_{i=k+1}^{n} X_{i:n}.$$

This is given in Theorem 2.

The distribution of Y_{n-k} turns out to involve only a k-fold integration, and so the numerical task is minimal for small values of k. As we are primarily interested in exact answers, for small values of n and k, Theorem 2 is entirely adequate for our purposes. Exact results for functions of order statistics from non-identically distributed variables are quite rare, a situation which was noted by David [1] and has evidently not changed significantly in the interim. Thus, this work may be considered a contribution to a significant gap in the literature.

Despite our focus on exact results, nonetheless, it is also of interest to obtain asymptotic approximations to the distribution, so as to establish ultimate limits to performance when large numbers of antennae are used. For this reason, we also give a normal limit, valid as $n \to \infty$ with k remaining fixed, and assess its accuracy. Comparisons are made as well with the asymptotic distribution as both n and $k \to \infty$ with $k/n \to \alpha > 0$. This latter approximation is in very close agreement with the exact values, even for $k = 1$.

All proofs are in the appendix.

2. Exact results

Before exhibiting the distribution of Y_{n-k}, we require a preliminary result. It serves to reduce an n-dimensional problem to a k-dimensional one. For $k = 1$, the result was apparently first used, in another form, by Dunnett and Sobel [2] – see [3, 4] – but for $k > 1$ appears to be new.

We write $(\mathbf{i})$ for a generic sequence $(i_1, \dots, i_k)$ with $1 \le i_1 < \cdots < i_k \le n$. Similarly $[\mathbf{i}]$ is the ordered $(n-k)$-tuple of indices $i_l \notin (\mathbf{i})$. In this notation,

$$Y_{n-k} = \max_{[\mathbf{i}]} \sum_{l \in [\mathbf{i}]} X_l. \tag{1}$$

LEMMA 1 *Let $\{X_1, \dots, X_n\}$ and $\{U_0, U_1, \dots, U_n\}$ be independent $N(0, 1)$ samples. Then, the $\binom{n}{k}$ random variables $\{\sum_{l \in [\mathbf{i}]} X_l\}$ are distributed in the same manner as the $\binom{n}{k}$ random variables $\{\sum_{l \in (\mathbf{i})} U_l + \sqrt{n-2k}\, U_0\}$.*

To express the distribution of Y_{n-k}, we denote by $\phi(\cdot)$ and $\Phi(\cdot)$ the density and cumulative distribution function of a $N(0, 1)$ random variable. Let $V_{j:n}$ be the jth smallest of the random variables $\{V_i = U_i - (\mu_i/\sigma)\}_{i=1}^{n}$, where $U_1, \dots, U_n \overset{i.i.d.}{\sim} N(0, 1)$, and denote by $H_k(v; \mu, \sigma)$ the distribution function of $\sum_{j=n-k+1}^{n} V_{j:n}$.

THEOREM 2 *Define $M_n = \sum_{i=1}^{n} \mu_i$. The distribution function of Y_{n-k} is*

$$F_{n-k}(y) = \int_{-\infty}^{\infty} H_k\left(\frac{y - M_n}{\sigma} - \sqrt{n-2k}\, u; \mu, \sigma\right) \phi(u)\mathrm{d}u, \tag{2}$$

and hence that of the normalized random variable $Z_{n-k} = (Y_{n-k} - M_n)/\sigma\sqrt{n-2k}$ is

$$G_{n-k}(z) = \int_{-\infty}^{\infty} H_k\left(\sqrt{n-2k}\,(z-u)\,; \mu, \sigma\right) \phi(u)\mathrm{d}u. \tag{3}$$

To apply Theorem 2, we require the distributions H_k. These are given by the following result.

LEMMA 3 *The distributions $H_k(v;\mu,\sigma)$ of $\sum_{j=n-k+1}^{n} V_{j:n}$ are given by*

$$H_1(v;\mu,\sigma)=\prod_{j=1}^{n}\Phi\left(v+\frac{\mu_j}{\sigma}\right), \tag{4}$$

and for $k\geq 2$

$$H_k(v;\mu,\sigma)=\int_{-\infty}^{v/k}H_1(u;\mu,\sigma)\sum_{(\mathbf{j})}\sum_{l=1}^{k}\psi_l\,(u;j_1,\ldots,j_k)\,\mathrm{d}u, \tag{5}$$

where

$$\begin{aligned}\psi_l(u;j_1,\ldots,j_k)=&\frac{\phi(u+(\mu_{j_l}/\sigma))}{\Phi(u+(\mu_{j_l}/\sigma))}\int_u^{v-(k-1)u}\frac{\phi(u_k+(\mu_{j_k}/\sigma))}{\Phi(u+(\mu_{j_k}/\sigma))}\cdots\\&\times\int_u^{v-lu-\sum_{i>l+1}u_i}\frac{\phi(u_{l+1}+(\mu_{j_{l+1}}/\sigma))}{\Phi(u+(\mu_{j_{l+1}}/\sigma))}\\&\times\int_u^{v-(l-1)u-\sum_{i>l-1,i\neq l}u_i}\frac{\phi(u_{l-1}+(\mu_{j_{l-1}}/\sigma))}{\Phi(u+(\mu_{j_{l-1}}/\sigma))}\cdots\\&\times\int_u^{v-u-\sum_{i>1,i\neq l}u_i}\frac{\phi(u_1+(\mu_{j_1}/\sigma))}{\Phi(u+(\mu_{j_1}/\sigma))}\mathrm{d}u_1\cdots\mathrm{d}u_{l-1}\,\mathrm{d}u_{l+1}\cdots\mathrm{d}u_k.\end{aligned} \tag{6}$$

In particular,

$$H_2(v;\mu,\sigma)=H_1\left(\frac{v}{2};\mu,\sigma\right)+\int_{-\infty}^{v/2}H_1(u;\mu,\sigma)\left[\sum_{j=1}^{n}\frac{\phi(v-u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\right]\mathrm{d}u, \tag{7}$$

$$\begin{aligned}H_3(v;\mu,\sigma)=&H_1\left(\frac{v}{3};\mu,\sigma\right)+\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\left[\sum_{j=1}^{n}\frac{2\phi(v-2u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\right]\mathrm{d}u,\\&+\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\int_u^{v-2u}\left[\sum_{i<j}\frac{\phi(t+(\mu_i/\sigma))}{\Phi(u+(\mu_i/\sigma))}\frac{\phi(v-u-t+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\right]\mathrm{d}t\,\mathrm{d}u\end{aligned} \tag{8}$$

In the special case $\mu_1=\cdots=\mu_n=\mu$, these become $H_1(v;\mu,\sigma)=\Phi^n(v+(\mu/\sigma))$ and

$$H_2(v;\mu,\sigma)=\Phi^n\left(\frac{v}{2}+\frac{\mu}{\sigma}\right)+n\int_{-\infty}^{v/2}\Phi^{n-1}\left(u+\frac{\mu}{\sigma}\right)\phi\left(v-u+\frac{\mu}{\sigma}\right)\mathrm{d}u,$$

$$\begin{aligned}H_3(v;\mu,\sigma)=&\Phi^n\left(\frac{v}{3}+\frac{\mu}{\sigma}\right)+2n\int_{-\infty}^{v/3}\Phi^{n-1}\left(u+\frac{\mu}{\sigma}\right)\phi\left(v-2u+\frac{\mu}{\sigma}\right)\mathrm{d}u\\&+\frac{n(n-1)}{2}\int_{-\infty}^{v/3}\Phi^{n-2}\left(u+\frac{\mu}{\sigma}\right)\int_u^{v-2u}\phi\left(t+\frac{\mu}{\sigma}\right)\phi\left(v-u-t+\frac{\mu}{\sigma}\right)\mathrm{d}t\,\mathrm{d}u.\end{aligned}$$

3. Asymptotics

The exact results of the previous section are suitable for small values of k but soon become quite intractable. In this section, we discuss and compare some asymptotic approximations.

An approximation to $G_{n-k}(z)$ as $n \to \infty$ follows from equation (3) by which we can represent Z_{n-k} as

$$Z_{n-k} = U_0 + \frac{\sum_{i=n-k+1}^{n} V_{i:n}}{\sqrt{n-2k}}.$$

Make the definitions

$$c_{k,n}^{\max} = \frac{\sum_{i=n-k+1}^{n} \mu_{i:n}}{\sigma\sqrt{n-2k}}, \quad c_{k,n}^{\min} = \frac{\sum_{i=1}^{k} \mu_{i:n}}{\sigma\sqrt{n-2k}},$$

and let $U_{i:n}$ be the ith-order statistic from $U_1, \ldots, U_n$. Then, we have

$$U_0 + \frac{\sum_{i=n-k+1}^{n} U_{i:n}}{\sqrt{n-2k}} - c_{k,n}^{\max} \leq Z_{n-k} \leq U_0 + \frac{\sum_{i=n-k+1}^{n} U_{i:n}}{\sqrt{n-2k}} - c_{k,n}^{\min}. \tag{9}$$

Cramér [5, p. 374] – see also Serfling [6, p. 90] – established that $U_{n:n} = o_p(\sqrt{n})$, so that $U_{i:n} = o_p(\sqrt{n})$ for each i. Then, $\sum_{i=n-k+1}^{n} U_{i:n}/\sqrt{n-2k} \overset{pr}{\to} 0$. As long as $c_{k,n}^{\max}$ and $c_{k,n}^{\min} \to 0$, or equivalently $\mu_{1:n}, \mu_{n:n} = o(\sqrt{n})$, we have that

$$Z_{n-k} \overset{\mathcal{L}}{\to} N(0, 1). \tag{10}$$

The denominator $\sqrt{n-2k}$ of Z_{n-k} may of course be replaced by the more conventional $\sqrt{n}$. The former is used here because it arises in a natural fashion and also to account for the greater variation expected as k increases.

Remark Let U, V be i.i.d. $N(0, 1)$ variables and define $\Lambda_n(u; t) = P(V \leq (z-u)\sqrt{n-2} + t)$. Then,

$$\begin{aligned} G_{n-1}(z) &= E\left[\prod_{i=1}^{n} \Lambda_n\left(U; \frac{\mu_i}{\sigma}\right)\right] \geq \prod_{i=1}^{n} E\left[\Lambda_n\left(U; \frac{\mu_i}{\sigma}\right)\right] \\ &= \prod_{i=1}^{n} \Phi\left(z\sqrt{\frac{n-2}{n-1}} + \frac{\mu_i}{\sigma\sqrt{n-1}}\right). \end{aligned} \tag{11}$$

The inequality follows from the fact that $\Lambda_n(u; t)$ is bounded and non-decreasing in u. Dunnett and Sobel [2] study mixtures of integrals of form (2) and derive bounds on the mixtures through the use of bound (11) applied to $F_{n-1}(y)$. Alternate bounds, which follow from (9) and are also easily computed, are

$$\begin{aligned} &\int_{-\infty}^{\infty} \Phi^n\left(\sqrt{n-2}\left(z-u+c_n^{\min}\right)\right)\phi(u)\,\mathrm{d}u \leq G_{n-1}(z) \\ &\leq \int_{-\infty}^{\infty} \Phi^n\left(\sqrt{n-2}\left(z-u+c_n^{\max}\right)\right)\phi(u)\,\mathrm{d}u. \end{aligned} \tag{12}$$

When assessed numerically, neither equation (11) nor equation (12) resulted in approximations to $G_{n-1}(z)$ more accurate than equation (10).

Table 1. Values of $G_{n-k}(z)$ from equation (3), its L-approximation (14) and the normal approximation $\Phi(z)$, for various choices of z, n and signal-to-noise ratios $\{\mu_i/\sigma\}_{i=1}^n$.

		$\{\mu_i/\sigma\} = \{1, 2, 3\}, n = 3$				$\{\mu_i/\sigma\} = \{1, 2, 3, 4, 5\}, n = 5$			
k	z	0	1	2	3	0	1	2	3
1	Exact	0.7253	0.9107	0.9812	0.9974	0.6569	0.9027	0.9855	0.9989
1	L-approximation	0.6718	0.8879	0.9764	0.9971	0.6180	0.8855	0.9824	0.9987
1	$(\mu_\alpha, \sigma_\alpha)^\dagger$		(1.807, 0.7496)				(2.885, 0.8582)		
2	Exact‡	–	–	–	–	0.9523	0.9881	0.9978	0.9997
2	L-approximation	–	–	–	–	0.9416	0.9856	0.9975	0.9997
2	$(\mu_\alpha, \sigma_\alpha)^\dagger$	–	–	–	–		(2.491, 0.7252)		
	Normal	0.5000	0.8413	0.9772	0.9987	0.5000	0.8413	0.9772	0.9987

		$\{\mu_i/\sigma\} = \{1, 1.5, \ldots, 6\}, n = 11$				$\{\mu_i/\sigma\} = \{1, 1.1, \ldots, 10\}, n = 91$			
k	z	0	1	2	3	0	1	2	3
1	Exact	0.5652	0.8715	0.9822	0.9989	0.4910	0.8354	0.9758	0.9985
1	L-approximation	0.5363	0.8557	0.9789	0.9987	0.4824	0.8299	0.9745	0.9984
1	$(\mu_\alpha, \sigma_\alpha)^\dagger$		(3.474, 0.9324)				(5.505, 0.9910)		
2	Exact	0.7382	0.9393	0.9930	0.9996	0.5039	0.8425	0.9772	0.9986
2	L-approximation	0.7099	0.9289	0.9914	0.9995	0.4938	0.8363	0.9758	0.9985
2	$(\mu_\alpha, \sigma_\alpha)^\dagger$		(3.354, 0.8725)				(5.502, 0.9827)		
	Normal	0.5000	0.8413	0.9772	0.9987	0.5000	0.8413	0.9772	0.9987

†Asymptotic mean and standard deviation $\mu_\alpha\left(\bar{F}_n\right), \sigma_\alpha\left(\bar{F}_n\right)$, with $\alpha = k/n$. All comparisons use $\sigma = 1$.
$^\ddagger G_{n-k}$ is not defined if $n \leq 2k$.

A comparison of the exact values of $G_{n-k}(z)$, and its normal approximation as $n \to \infty$, is given in table 1 for $k = 1, 2$. For $k = 1$, the upper tail probabilities are fairly well approximated by the normal distribution for n as small as 3. For $n = 5$ and $n = 11$, see also figure 1. For $k = 2$, the approximation is less accurate unless n is quite large. The values of G_{n-1} and G_{n-2} were obtained by a direct evaluation of equations (3), (4) and (7), using Simpson's rule to approximate the integrals. An S-Plus programme to compute these values, and the others which appear in this article, is posted at http://www.stat.ualberta.ca/∼wiens/reprints/order.ssc.

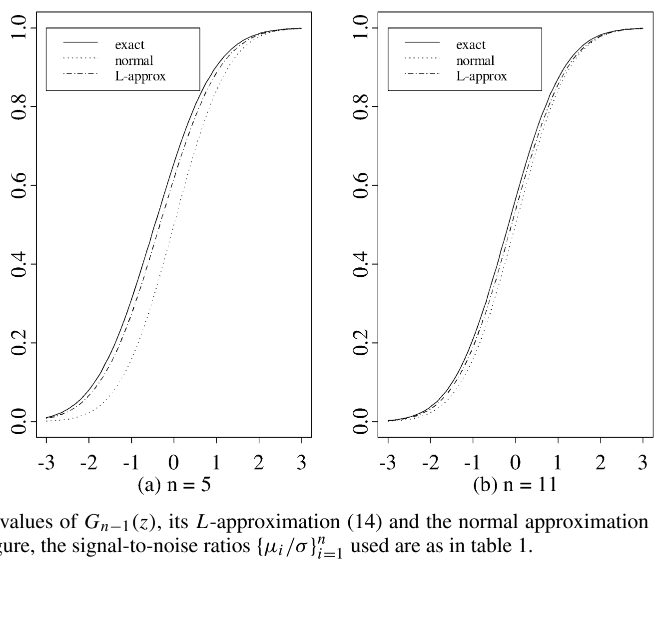

Figure 1. Exact values of $G_{n-1}(z)$, its L-approximation (14) and the normal approximation $\Phi(z)$ for $n = 5$ and $n = 11$. In each figure, the signal-to-noise ratios $\{\mu_i/\sigma\}_{i=1}^n$ used are as in table 1.

If both n and $k \to \infty$ with $k/n \to \alpha > 0$, then the results of Stigler [7], on L-statistics for independent but not identically distributed observations, hold. To apply these results, let $\bar{F}_n(x) = n^{-1}\sum_{i=1}^{n}\Phi((x-\mu_i)/\sigma)$ be the average distribution and define

$$K_n(x, y) = \bar{F}_n(\min(x, y)) - \frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{x-\mu_i}{\sigma}\right)\Phi\left(\frac{y-\mu_i}{\sigma}\right),$$

$$J_\alpha(t) = \begin{cases} 1, & \alpha \le t \le 1, \\ 0, & 0 \le t < \alpha. \end{cases}$$

Then, by Theorem 6 of Stigler [7], if the sequence $\{\mu_i\}$ is bounded, we have that

$$\sqrt{n}\left(\frac{(Y_{n-k}/n) - \mu_\alpha(\bar{F}_n)}{\sigma_\alpha(\bar{F}_n)}\right) \xrightarrow{\mathcal{L}} N(0, 1), \tag{13}$$

for

$$\mu_\alpha(\bar{F}_n) = \int_0^1 J_\alpha(u)\bar{F}_n^{-1}(u)\mathrm{d}u = \int_{\bar{F}_n^{-1}(\alpha)}^{\infty} t\,\mathrm{d}\bar{F}_n(t),$$

$$\sigma_\alpha^2(\bar{F}_n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_\alpha(\bar{F}_n(x))\, J_\alpha(\bar{F}_n(y))\, K_n(x, y)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{\bar{F}_n^{-1}(\alpha)}^{\infty}\int_{\bar{F}_n^{-1}(\alpha)}^{\infty} K_n(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

For purposes of comparison with equation (10), we note that equation (13) results in the 'L-approximation'

$$G_{n-k}(z) \approx \Phi\left(z\frac{\sigma}{\sigma_\alpha(\bar{F}_n)}\sqrt{1-\frac{2k}{n}} - \frac{\mu_\alpha(\bar{F}_n) - (M_n/n)}{\sigma_\alpha(\bar{F}_n)/\sqrt{n}}\right), \tag{14}$$

with α approximated by k/n. Some numerical values are in table 1 (see also figure 1). For $k = 1, 2$, there is excellent agreement between equation (14) and the exact values of G_{n-k}.

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Appendix

Derivations

Proof of Lemma 1 Both sets of random variables are jointly normally distributed, with common mean 0 and common variance $n-k$. Thus, it suffices to show that the covariances agree. The covariance between $\sum_{l\in[\mathbf{i}]} X_l$ and $\sum_{l\in[\mathbf{j}]} X_l$ is $\#\{[\mathbf{i}] \cap [\,\mathbf{j}]\}$. This cardinality is in turn equal to $n - 2k + \#\{(\mathbf{i}) \cap (\,\mathbf{j})\}$, which is the covariance between $\sum_{l\in(\mathbf{i})} U_l + \sqrt{n-2k}\,U_0$ and $\sum_{l\in(\,\mathbf{j})} U_l + \sqrt{n-2k}\,U_0$. ∎

Proof of Theorem 2 From equation (1) and Lemma 1, we have

$$\begin{aligned}
F_{n-k}(y) &= P\left(\bigcap_{[\mathbf{i}]}\left\{\sum_{l\in[\mathbf{i}]}\left(\frac{X_l-\mu_l}{\sigma}\right)+\sum_{l\in[\mathbf{i}]}\frac{\mu_l}{\sigma}\le\frac{y}{\sigma}\right\}\right)\\
&= P\left(\bigcap_{(\mathbf{i})}\left\{\sum_{l\in(\mathbf{i})}U_l+\sqrt{n-2k}\,U_0+\sum_{l\in[\mathbf{i}]}\frac{\mu_l}{\sigma}\le\frac{y}{\sigma}\right\}\right)\\
&= P\left(\bigcap_{(\mathbf{i})}\left\{\sum_{l\in(\mathbf{i})}\left(U_l-\frac{\mu_l}{\sigma}\right)+\sqrt{n-2k}\,U_0\le\frac{y-M_n}{\sigma}\right\}\right)\\
&= P\left(\max_{(\mathbf{i})}\sum_{l\in(\mathbf{i})}\left(U_l-\frac{\mu_l}{\sigma}\right)\le\frac{y-M_n}{\sigma}-\sqrt{n-2k}\,U_0\right)\\
&= P\left(\sum_{j=n-k+1}^{n}V_{j:n}\le\frac{y-M_n}{\sigma}-\sqrt{n-2k}\,U_0\right).
\end{aligned}$$

The result now follows by conditioning on U_0. ■

Proof of Lemma 3 For $k=1$, H_k is merely the distribution of the largest order statistic given by equation (4). In general, H_k can be obtained by an appropriate integration over the density of $\{V_{j:n}\}_{j=n-k+1}^{n}$. This density is

$$\begin{aligned}
\xi(u_1,\ldots,u_k) &= \sum_{(\mathbf{j})}\prod_{l\notin\{j_1,\ldots,j_k\}}\Phi\left(\min(u_1,\ldots,u_k)+\frac{\mu_l}{\sigma}\right)\prod_{i\le k}\phi\left(u_i+\frac{\mu_{j_i}}{\sigma}\right)\\
&= H_1(\min(u_1,\ldots,u_k);\mu,\sigma)\sum_{(\mathbf{j})}\prod_{i\le k}\frac{\phi(u_i+(\mu_{j_i}/\sigma))}{\Phi(\min(u_1,\ldots,u_k)+(\mu_{j_i}/\sigma))}
\end{aligned}$$

and is to be integrated over $\cup_{l=1}^{k}R(u_l)$, where $R(u_l)=\{u_i\ge u_l(i\ne l),\sum_{i\ne l}u_i\le v-u_l\}$ for $u_l\le v/k$. Thus,

$$\begin{aligned}
H_k(v;\mu,\sigma) &= \sum_{l=1}^{k}\int_{-\infty}^{v/k}\underbrace{\int\cdots\int}_{\cup_{l=1}^{k}R(u_l)}H_1(u_l;\mu,\sigma)\\
&\quad\times\sum_{(\mathbf{j})}\prod_{i\le k}\frac{\phi(u_i+(\mu_{j_i}/\sigma))}{\Phi(u_l+(\mu_{j_i}/\sigma))}\mathrm{d}u_1\cdots\mathrm{d}u_{l-1}\,\mathrm{d}u_{l+1}\cdots\mathrm{d}u_k\,\mathrm{d}u_l,
\end{aligned}$$

and this becomes equation (5) for

$$\begin{aligned}
&\psi_l(u;j_1,\ldots,j_k)\\
&\quad=\frac{\phi(u+(\mu_{j_l}/\sigma))}{\Phi(u+(\mu_{j_l}/\sigma))}\underbrace{\int\cdots\int}_{\cup_{l=1}^{k}R(u_l)}\prod_{i\le k,i\ne l}\frac{\phi(u_i+(\mu_{j_i}/\sigma))}{\Phi(u+(\mu_{j_i}/\sigma))}\mathrm{d}u_1\cdots\mathrm{d}u_{l-1}\,\mathrm{d}u_{l+1}\cdots\mathrm{d}u_k.
\end{aligned}$$

■

Expressing ψ_l as an iterated integral gives equation (6).

For $k = 2$, this procedure yields

$$H_2(v;\mu,\sigma) = \int_{-\infty}^{v/2} H_1(u;\mu,\sigma)\sum_{i<j}\left\{\frac{\phi(u+(\mu_i/\sigma))}{\Phi(u+(\mu_i/\sigma))}\int_u^{v-u}\frac{\phi(u_2+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\mathrm{d}u_2 + \frac{\phi(u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\int_u^{v-u}\frac{\phi(u_1+(\mu_i/\sigma))}{\Phi(u+(\mu_i/\sigma))}\mathrm{d}u_1\right\}\mathrm{d}u$$
$$= \int_{-\infty}^{v/2} H_1(u;\mu,\sigma)\sum_{i\neq j}\frac{\phi(u+(\mu_i/\sigma))}{\Phi(u+(\mu_i/\sigma))}\left\{\frac{\Phi(v-u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}-1\right\}\mathrm{d}u. \quad \text{(A1)}$$

As $H_1(u;\mu,\sigma)$ has density

$$h_1(u;\mu,\sigma) = H_1(u;\mu,\sigma)\sum_{j=1}^{n}\frac{\phi(u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))},$$

equation (A1) can be continued as

$$H_2(v;\mu,\sigma) = \sum_{j=1}^{n}\int_{-\infty}^{v/2}\left[\frac{\Phi(v-u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}-1\right] \times\left[h_1(u;\mu,\sigma) - H_1(u;\mu,\sigma)\frac{\phi(u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\right]\mathrm{d}u.$$

Evaluating $\int_{-\infty}^{v/2}[(\Phi(v-u+(\mu_j/\sigma))/\Phi(u+(\mu_j/\sigma)))-1]h_1(u;\mu,\sigma)\mathrm{d}u$ by parts and simplifying yields

$$H_2(v;\mu,\sigma) = \sum_{j=1}^{n}\int_{-\infty}^{v/2} H_1(u;\mu,\sigma)\left[\frac{\phi(v-u+(\mu_j/\sigma))+\phi(u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\right]\mathrm{d}u$$
$$= \sum_{j=1}^{n}\int_{-\infty}^{v/2} H_1(u;\mu,\sigma)\left[\frac{\phi(v-u+(\mu_j/\sigma))}{\Phi(u+(\mu_j/\sigma))}\right]\mathrm{d}u + \int_{-\infty}^{v/2}\mathrm{d}H_1(u;\mu,\sigma),$$

which is equation (7).

A similar but somewhat lengthy calculation, the details of which are available upon request, gives equation (8).

There is an alternate and much simpler derivation of equation (7) which unfortunately does not seem to generalize. First note that $H_2(v;\mu,\sigma) = P(V_{n-1:n}+V_{n:n}\le v)$. We decompose the event $E = \{V_{n-1:n}+V_{n:n}\le v\}$ according to whether $V_{n:n}\le v/2$ (implying E) or $V_{n:n} > v/2$ (requiring $n-1$ of the V_j to be $\le v/2$). Thus,

$$H_2(v;\mu,\sigma) = P\left(V_{n:n}\le\frac{v}{2}\right) + P\left(\left\{V_{n:n}>\frac{v}{2}\right\}\cap\{V_{n-1:n}+V_{n:n}\le v\}\right)$$
$$= H_1\left(\frac{v}{2};\mu,\sigma\right) + \sum_{j=1}^{n}\int_{v/2}^{\infty}\prod_{i\neq j}P(V_i\le v-w)\mathrm{d}P(V_j\le w)$$
$$= H_1\left(\frac{v}{2};\mu,\sigma\right) + \sum_{j=1}^{n}\int_{v/2}^{\infty}H_1(v-w;\mu,\sigma)\frac{\phi(w+(\mu_j/\sigma))}{\Phi(v-w+(\mu_j/\sigma))}\mathrm{d}w,$$

which gives equation (7) upon making the change of variable $u = v - w$ in the final integral.

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