PROBLEMS

02.1.1. LS and BLUE Are Algebraically Identical, proposed by Richard W. Farebrother. Let $X$ be a given $n \times q$ matrix of rank $q$, let $y$ be a given $n \times 1$ matrix, and let $b$ be a $q \times 1$ matrix of unknown elements. Then the least squares problem is defined by:

$$\min e' e \text{ subject to } e = y - Xb.$$ 

Further, let $c \neq 0$ be a given $q \times 1$ matrix and let $a$ be an $n \times 1$ matrix of unknown elements. Then the minimum variance unbiased linear estimation problem is defined by:

$$\min a' a \text{ subject to } X'a = c.$$ 

It is well known that the solutions $a^* = X(X'X)^{-1}c$ and $b^* = (X'X)^{-1}X'y$ to these problems are connected by the relationship $a''y = c'b^*$. Readers are invited to show that the problems themselves are algebraically identical.

02.1.2. A Particular Symmetric Idempotent Matrix, proposed by Heinz Neudecker. Consider the $q \times q$ symmetric real matrices $B$ and $C = \begin{pmatrix} rI_m & 0 \\ 0 & 0 \end{pmatrix}$, $m < q$, integer $r > 1$. Let $tr(B^k) = tr(C^k)$, $k = 1,2,3,\ldots$. Show that $(1/r)B$ is idempotent.

CORRIGENDUM

Geert Dhaene has pointed out that his solution to Problem 00.1.1’, Determinant of a Skew-Symmetric Matrix, published in Econometric Theory 17, pp. 277–278, contains an error. A correction by the author is the following.

By the skew-symmetry of $S$, we have $|S_{j1,1i}| = (-1)^{a-2}|S_{11,1j}|$ and

$$|S_{j1,1j}| + |S_{j1,1i}| = \begin{cases} 2|S_{i1,1j}| & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$
Therefore, the statement \(|S| = \sum_{i=2}^{k} s_{i}^{2} |s_{i,11}|\) at the beginning of page 278 (and the proof) is correct only when \(n\) is odd. A correct solution of the problem when \(n\) is even is as follows. Consider the \(n \times n\) skew-symmetric matrix

\[
S = \begin{pmatrix}
S_{11} & -S'_{21} \\
S_{21} & S_{22}
\end{pmatrix}, \quad \text{where } S_{22} = \begin{pmatrix}
0 & -s_{21} \\
s_{21} & 0
\end{pmatrix}
\]

and \(s_{21}\) is a scalar. \(|S_{22}| \geq 0\). Assume that skew-symmetric matrices of even order up to \((n-2) \times (n-2)\) have a nonnegative determinant. Then, if \(s_{21} \neq 0\),

\[
|S| = |S_{22}| |S_{11} + S'_{21} S_{22}^{-1} S_{21}| \geq 0,
\]

since \(S_{11} + S'_{21} S_{22}^{-1} S_{21}\) is skew-symmetric. Moreover \(|S| \geq 0\) also if \(s_{21} = 0\), by the continuity of \(|S|\) in \(s_{21}\). The result follows by induction.
PROBLEMS AND SOLUTIONS

PROBLEMS

03.1.1. Deriving the Observed Information Matrix in Ordered Probit and Logit Models Using the Complete-Data Likelihood Function

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Louis (1982) presents a method for computing the observed information matrix and standard errors of maximum likelihood estimates obtained via the EM algorithm based on the complete-data log likelihood function. The problem illustrates the well-known method of Louis (1982) for a widely used qualitative response model in econometrics. The observed-data log likelihood function for the following model can, of course, be easily differentiated to obtain the observed information matrix; our objective is to illustrate the method and not to recommend its use for this model.

Consider the following ordered response model:

\[ y^*_i = \beta' x_i + u_i, \quad i = 1, 2, \ldots, n, \tag{1} \]

where \( y^* \) is the underlying response variable, \( \beta \) is a \((k \times 1)\) vector of unknown parameters, \( x \) is a \((k \times 1)\) vector of known constants, and \( u \) is the residual. The term \( y^* \) is not observed, but we know which of the \( m \) categories it belongs to. It belongs to the \( j \)th category if \( \alpha_{j-1} < y^* < \alpha_j \) (\( j = 1, 2, \ldots, m \)), \( \alpha \)'s being known constants and \( \alpha_0 = -\infty, \alpha_m = \infty \). Derive the observed information matrix for the model using the complete-data log likelihood function under the assumption that \( u_i \) are i.i.d. \( N(0,1) \).

REFERENCE


03.1.2. Redundancy of Lagged Regressors in a Conditionally Heteroskedastic Time Series Regression

Stanislav Anatolyev
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Consider the following stationary time series regression:

\[ y_t = \beta x_t + e_t, \quad E[e_t|x_t, x_{t-1}, \ldots] = 0, \]

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where all variables are scalars and the error $e_t$ is conditionally heteroskedastic with the following form of heteroskedasticity:

$$E[e_t^2|x_t, x_{t-1}, \ldots] = \omega + \lambda(x_t - \mu)^2, \quad \omega > 0, \quad \lambda \geq 0.$$ 

The object of estimation is $\beta$. Usually, under conditional heteroskedasticity the use of lagged values of regressors as instruments increases the efficiency of generalized method of moments (GMM) estimation in comparison with ordinary least squares (OLS) estimation (see, e.g., Broze, Francq, and Zakoian, 2001; West, 2001). This problem shows that it may not necessarily be so.

Assume that the regressor $x_t$ can be represented as $x_t = \sum_{i=0}^{\infty} \varphi_i \eta_{t-i}$, where $\eta_i$’s are independently and identically distributed standard normal. Also assume that the parameters are constrained so that all variables have finite fourth moments. Show that the OLS estimator is at least as efficient as any GMM estimator that uses an arbitrary fixed number of instruments from the list \{x_t, x_{t-1}, x_{t-2}, \ldots\}.

REFERENCES


SOLUTIONS

02.1.1. LS and BLUE Are Algebraically Identical—Solution

Richard William Farebrother (the poser of the problem)

*Shrewsbury, UK*

Let $Z$ be an $n \times (n - q)$ matrix of rank $n - q$ whose columns span the space orthogonal to the space spanned by the columns of $X$ and let $f = Z' y$. Then, on premultiplying by the $n \times n$ nonsingular matrix $[Z:X]$, we find that the conventional least squares problem:

$$\min e'e \quad \text{subject to } e = y - Xb$$

may be expressed as

$$\min e'e \quad \text{subject to } Z'e = Z'y \quad \text{and } X'Xb = X'y - X'e$$

or, equivalently, as

$$\min e'e \quad \text{subject to } Z'e = f$$

and

$$b = (X'X)^{-1}X'(y - e),$$
where equation (4) defines the optimal value of $b$ in terms of the optimal value of $e$.

Further, let $u$ be an $n \times 1$ matrix satisfying $X'u = c$ and let $d = (Z'Z)^{-1}Z'(a - u)$ be an $(n - q) \times 1$ function of $a$. Then the conventional minimum variance unbiased linear estimation problem

$$\min a'a \quad \text{subject to} \quad X'a = c$$

may be expressed as

$$\min a'a \quad \text{subject to} \quad X'a = X'u - X'Zd \quad \text{and} \quad Z'a = Z'u - Z'Zd,$$

and, on premultiplying by the inverse of the $n \times n$ nonsingular matrix $[Z: X]$, we have

$$\min a'a \quad \text{subject to} \quad a = u - Zd.$$  

Clearly, problems (1) and (3) are algebraically identical to problems (7) and (5) when $X, y, b, f, and e$ are replaced by $Z, u, d, c$, and $a$, respectively.

**NOTE**

1. Two excellent solutions were independently proposed by H. Neudecker and S. Puntanen, G. Styan, and H. Werner.

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## 02.1.2. A Particular Symmetric Idempotent Matrix—Solution

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Hans Joachim Werner  
*University of Bonn*

It is easy to see that Problem 02.1.2 holds more generally in that the result is valid for $B$ Hermitian complex (rather than “symmetric real”), for $m = 0,1,\ldots,q$ (rather than just for “$m < q$”), for $r \neq 0$ (rather than just for “integer $r \geq 1$”), and for $\text{tr}(B^k) = \text{tr}(C^k)$, $k = 1,2,3,4$ (rather than for “$k = 1,2,\ldots$”), as shown by Shanbhag (1970).

To see this let $A = (1/r)B$, $D = (1/r)C$, and $\alpha_1, \alpha_2,\ldots, \alpha_q$ denote the eigenvalues of $A$. Then

$$m = \sum_{i=1}^{q} \alpha_i = \sum_{i=1}^{q} \alpha_i^2 = \sum_{i=1}^{q} \alpha_i^3 = \sum_{i=1}^{q} \alpha_i^4,$$

because we know that if $\lambda_1,\ldots, \lambda_q$ are the (not necessarily distinct) eigenvalues of $M \in \mathbb{C}^{q \times q}$, then $\lambda_1^n,\ldots, \lambda_q^n$ are the eigenvalues of $M^n$, irrespective of $n \in \mathbb{N}$ (see, e.g., Lancaster, 1969, Theorem 2.5.2). A fundamental result on characteristic polynomials tells us that $\text{tr}(M) = \sum_{i=1}^{q} \lambda_i$ (cf. Lancaster, 1969, p. 55). Hence
\[ \sum_{i=1}^{q} \alpha_i^2(1 - \alpha_i)^2 = 0, \]

and so the \( \alpha_i = 0 \) or 1. Because \( \sum_{i=1}^{q} \alpha_i = m \) it follows that precisely \( m \) of the \( \alpha_i \) are equal to 1 and \( q - m \) are equal to 0. A Hermitian matrix is unitarily similar to the diagonal matrix of its eigenvalues, and the eigenvalues are all real (cf. e.g., Lancaster, 1969, Theorem 2.9.5 and Theorem 2.9.1). It follows then, as is well known, that a Hermitian complex matrix with all eigenvalues equal to 0 or 1 is idempotent. Our proof is complete.

One might be tempted to believe that a complex square matrix \( M \), not Hermitian, with all eigenvalues equal to 0 and/or 1 is always idempotent. This, however, is erroneous, as the nilpotent matrix

\[
M = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

shows. Although this matrix fails to be idempotent, it has the eigenvalue 0 with (algebraic) multiplicity 2.

When \( A \) and \( B \) are both Hermitian nonnegative definite, then the eigenvalues \( \alpha_i \geq 0 \), and, as Shanbhag (1970) points out, the condition \( \text{tr}(B^k) = \text{tr}(C^k) \), \( k = 1,2,3,4 \) may be replaced by \( \text{tr}(B^k) = \text{tr}(C^k) \), \( k = 1,2,3 \). This leads to \( \sum_{i=1}^{q} \alpha_i(1 - \alpha_i)^2 = 0 \), and because the \( \alpha_i \) are now all nonnegative we see immediately that \( \alpha_i = 0 \) or 1.

**NOTE**

1. Excellent solutions were independently proposed by G. Dhaene, J. Hill, S. Lawford and C. Dehon, P. Omtzigt, G. Trenkler, M. Van de Velden, D. Wiens, and H. Neudecker (the poser of the problem).

**REFERENCES**


Solution to Problem 02.1.2 in *Econometric Theory* 18, 193-194.
Doug Wiens
University of Alberta

Define \( A = (1/r)B \), a \( q \times q \) real, symmetric matrix. We are to show that \( A \) is idempotent, or equivalently that all of its eigenvalues are in \( \{0, 1\} \).

The condition \( tr(B^k) = tr(C^k) \) is equivalent to

\[
tr(A^k) = m < q, \text{ for all } k = 1, 2, \ldots .
\]

In terms of the eigenvalues \( \lambda_1, \ldots, \lambda_q \) of \( A \) this condition is

\[
\frac{1}{q} \sum_{i=1}^{q} \lambda_i^k = p := \frac{m}{q} < 1, \text{ for all } k = 1, 2, \ldots . \tag{1}
\]

Define a r.v. \( \Lambda \) to be equal to \( \lambda_i \) with probability \( 1/q \), for \( i = 1, \ldots, q \). We must show that

\[
P(\Lambda \in \{0, 1\}) = 1. \tag{2}
\]

Then all \( \lambda_i \in \{0, 1\} \), as claimed.

Condition (1) states that all integral moments \( E[\Lambda^k] \) equal \( p \). One distribution with these moments is the Bernoulli with “success” probability \( p \); this distribution clearly satisfies (2). To show that this is the only such distribution it suffices to verify Carleman’s condition. This is immediate:

\[
\sum_{r=1}^{\infty} \left( \frac{1}{E[\Lambda^{2r}]} \right)^{1/2r} = \sum_{r=1}^{\infty} \left( \frac{1}{p} \right)^{1/2r} = \infty,
\]

since \((1/p)^{1/2r} \to 0\) as \( r \to \infty \).