1. If \( f \) and \( g \) are twice differentiable functions of a single variable, show that the function

\[ u(x, y) = x f(x + y) + y g(x + y) \]

satisfies the equation \( u_{xx} - 2u_{xy} + u_{yy} = 0 \).

**Solution**

If we let \( s = x + y \) we can write \( u = x f(s) + y g(s) \). Then using the chain rule, and the fact that \( \frac{\partial s}{\partial x} = \frac{\partial s}{\partial y} = 1 \), we get

\[
\begin{align*}
    u_x &= f(s) + x \frac{df}{ds} \frac{\partial s}{\partial x} + y \frac{dg}{ds} \frac{\partial s}{\partial x} = f(s) + x f'(s) + y g'(s), \\
    u_y &= x \frac{df}{ds} \frac{\partial s}{\partial y} + g(s) + y \frac{dg}{ds} \frac{\partial s}{\partial y} = x f'(s) + g(s) + y g'(s), \\
    u_{xx} &= \frac{df}{ds} \frac{\partial s}{\partial x} + f'(s) + x \frac{d[f'(s)]}{ds} \frac{\partial s}{\partial x} + y \frac{d[g'(s)]}{ds} \frac{\partial s}{\partial x} = 2f'(s) + x f''(s) + y g''(s), \\
    u_{xy} &= \frac{df}{ds} \frac{\partial s}{\partial y} + x \frac{d[f'(s)]}{ds} \frac{\partial s}{\partial y} + g'(s) + y \frac{d[g'(s)]}{ds} \frac{\partial s}{\partial y} = f'(s) + x f''(s) + g'(s) + y g''(s), \\
    u_{yy} &= x \frac{d[f'(s)]}{ds} \frac{\partial s}{\partial y} + \frac{dg}{ds} \frac{\partial s}{\partial y} + g'(s) + y \frac{d[g'(s)]}{ds} \frac{\partial s}{\partial y} = x f''(s) + 2g'(s) + y g''(s).
\end{align*}
\]

Using these expressions we see that

\[
\begin{align*}
    u_{xx} - 2u_{xy} + u_{yy} &= [2f'(s) + x f''(s) + yg''(s)] \\
    &- [2f'(s) + x f''(s) + g'(s) + yg''(s)] + [x f''(s) + 2g'(s) + yg''(s)] = 0.
\end{align*}
\]

2. If \( u = e^{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n} \), where \( a_1^2 + a_2^2 + \cdots + a_n^2 = 1 \), show that

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u.
\]

**Solution**

If we define \( s = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \) then we can write \( u = e^s \). Differentiating gives

\[
\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} e^s = \frac{d}{ds} (e^s) \frac{\partial s}{\partial x_i} = e^s \cdot a_i = a_i u, \quad \text{valid for } 1 \leq i \leq n.
\]

Differentiate again to get

\[
\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right) = a_i \frac{\partial u}{\partial x_i} = a_i^2 u.
\]

Summing these yields

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = a_1^2 u + a_2^2 u + \cdots + a_n^2 u = (a_1^2 + a_2^2 + \cdots + a_n^2) u = u.
\]
3. The partial differential equation\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]is called Laplace's Equation, named after the eminent French mathematician Pierre Simon de Laplace (1749 — 1827). Solutions of this equation are called harmonic functions and play a role in problems of heat conduction, fluid flow, and electric potential. Which of the following functions are solutions of Laplace's equation?

(a) \( u(x, y) = x^2 + y^2 \);
(b) \( u(x, y) = \ln((x^2 + y^2)^{3/2}) \);
(c) \( u(x, y) = \sin x \cosh y + \cos x \sinh y \).

**Solution**

(a) If \( u(x, y) = x^2 + y^2 \), then \( u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0 \). Therefore \( u(x, y) = x^2 + y^2 \) is not a solution.

(b) If \( u(x, y) = \ln((x^2 + y^2)^{3/2}) \), then we get
\[
\begin{align*}
u_x &= \frac{3x}{x^2 + y^2}, & u_y &= \frac{3y}{x^2 + y^2}, & u_{xx} &= \frac{3(-x^2 + y^2)}{(x^2 + y^2)^2}, & u_{yy} &= \frac{3(-x^2 + y^2)}{(x^2 + y^2)^2}.
\end{align*}
\]
Hence \( u_{xx} + u_{yy} = \frac{3(-x^2 + y^2 + x^2 - y^2)}{(x^2 + y^2)^2} = 0 \). Therefore \( u(x, y) = \ln((x^2 + y^2)^{3/2}) \) is a solution.

(c) If \( u(x, y) = \sin x \cosh y + \cos x \sinh y \), then
\[
\begin{align*}
u_{xx} &= -\sin x \cosh y - \cos x \sinh y, & u_{yy} &= \sin x \cosh y + \cos x \sinh y.
\end{align*}
\]
Hence \( u_{xx} + u_{yy} = -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y = 0 \).

Therefore \( u(x, y) = \sin x \cosh y + \cos x \sinh y \) is a solution.

4. Find the points on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 1 \) at which the tangent plane is parallel to the plane \( 3x - 2y + 3z = 1 \).

**Solution**

Let \( f(x, y, z) := x^2 + 2y^2 + 3z^3 \), so that the equation for the ellipsoid becomes \( f(x, y, z) = 1 \). A normal vector to the plane \( 3x - 2y + 3z = 1 \) is \( (3, -2, 3) \). A normal vector for the tangent plane at the point \( (x_0, y_0, z_0) \) on the ellipsoid is given by \( \nabla f(x_0, y_0, z_0) = (2x_0, 4y_0, 6z_0) \). Since the tangent plane is parallel to the given plane we must have \( \nabla f(x_0, y_0, z_0) = c (3, -2, 3) \). In other words we must have \( (2x_0, 4y_0, 6z_0) = c (3, -2, 3) \). Thus \( x_0 = 3c/2, y_0 = -c/2, \) and \( z_0 = c/2 \). But since \( (x_0, y_0, z_0) \) lies on the ellipsoid we have
\[
f(x_0, y_0, z_0) = 1 \quad \Rightarrow \quad \frac{14}{4} c^2 = 1 \quad \Rightarrow \quad c = \pm \frac{\sqrt{2}}{7}.
\]
Thus, there are two points at which the tangent plane for the ellipsoid is parallel to the given plane. These points are: \( \pm \frac{1}{\sqrt{14}} (3, -1, 1) \).
5. Find the directional derivative of the function at the point \( P \) in the direction of the vector \( \vec{v} \):

(a) \( f(x, y, z) = z^3 - x^2y \), \( P(1, 6, 2) \), \( \vec{v} = (3, 4, 12) \);

(b) \( g(x, y, z) = xe^{yz} + yze^x \), \( P(-2, 1, 1) \), \( \vec{v} = \hat{i} - 2\hat{j} + 3\hat{k} \).

**Solution**

(a) \( \vec{v}f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (-2xy, -x^2, 3z^2) \);

(b) \( \vec{v}g = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) = (e^{yz} + ye^z, xze^{yz} + xe^z, xy(e^{yz} + e^z)) \);

Thus \( D_{\vec{v}}f(1, 6, 2) = \nabla f(1, 6, 2) \cdot \vec{u} = 8 \).

6. Let \( f \) and \( g \) be two differentiable real valued functions of a single variable. Show that any function of the form \( z = f(x + at) + g(x - at) \) is a solution of the wave equation \( \frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \).

**Solution**

Let \( u = x + at \) and \( v = x - at \). Then \( z = f(u) + g(v) \) and the chain rule gives:

\[
\frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} + \frac{dg}{dv} \frac{\partial v}{\partial x} = \frac{df}{du} + \frac{dg}{dv}.
\]

Thus

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{df}{du} + \frac{dg}{dv} \right) = \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}.
\]

Combining these gives the desired result: \( \frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \).

7. Find the local maximum and minimum values and saddle point(s) of the function:

\( f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2 \).

**Solution**

The first and second partial derivatives of \( f \) are:

\( f_x(x, y) = 6xy - 6x \), \( f_y(x, y) = 3x^2 + 3y^2 - 6y \), \( f_{xx}(x, y) = 6y - 6 \), \( f_{xy}(x, y) = 6x \), \( f_{yy}(x, y) = 6y - 6 \).

First find the critical points:

\( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \) \( \implies \) \( (x, y) = (0, 0), (0, 2), (1, 1), (-1, 1) \).

To classify these points we need to examine \( f_{xx} \) and \( D = f_{xx}f_{yy} - f_{xy}^2 \). The results are summarized in the following table:
8. Find the extreme values of \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \) on the region \( D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 16\} \).

**Solution**
The first derivatives are given by \( f_x(x, y) = 4x - 4 \) and \( f_y(x, y) = 6y \). Now calculate the critical points:

\[ f_x(x, y) = 0 \quad \text{and} \quad f_y(x, y) = 0 \quad \implies \quad (x, y) = (1, 0). \]

The only critical point \((1, 0)\) lies in \( D \). At this point we have \( f(1, 0) = -7 \). On the boundary of \( D \) we have \( y^2 = 16 - x^2 \). Thus define:

\[ g(x) := f(x, \pm \sqrt{16 - x^2}) = -x^2 - 4x + 43, \quad \text{for} \ x \in [-4, 4]. \]

Finding critical points of \( g \) boundary we have

\[ g'(x) = 0 \quad \implies \quad -2x - 4 = 0 \quad \implies \quad x = -2 \quad \implies \quad y = \pm 2\sqrt{3}. \]

At these points we have \( f(-2, \pm 2\sqrt{3}) = 47 \). Checking the endpoints: \( g(-4) = 43 \) and \( g(4) = 11 \). Thus the maximum value of \( f \) on the disc \( D \) is 47, which occurs at the points \((-2, \pm 2\sqrt{3})\), and the minimum of \( f \) is \(-7\), which occurs at the point \((1, 0)\).

9. Find the absolute maximum and minimum values of \( f \) on the set \( D \).

(a) \( f(x, y) = y\sqrt{x} - y^2 - x + 6y, \) \( D = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq 9, 0 \leq y \leq 5\}; \)

(b) \( f(x, y) = 2x^2 + x + y^2 - 2, \) \( D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 4\} \).

**Solution**

(a) Calculating the partial derivatives gives:

\[ f_x = \frac{y}{2\sqrt{x}} - 1, \quad f_y = \sqrt{x} - 2y + 6. \]

Finding critical points: \( f_x = 0, \quad f_y = 0 \quad \implies \quad (x, y) = (4, 4). \)

Note that \((4, 4) \in D\) and that \( f(4, 4) = 12 \). The boundary of \( D \) consists of four line segments:

\[ L_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 9, \ y = 0\}, \quad L_2 = \{(x, y) \in \mathbb{R}^2 \mid x = 9, \quad 0 \leq y \leq 5\}, \]

\[ L_3 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 9, \ y = 5\}, \quad L_4 = \{(x, y) \in \mathbb{R}^2 \mid x = 0, \quad 0 \leq y \leq 5\}. \]

On \( L_1 \) we have \( f(x, 0) = -x \) which on \([0, 9]\) has a maximum value at \( x = 0: \ f(0, 0) = 0 \) and a minimum value at \( x = 9: \ f(9, 0) = -9 \).

On \( L_2 \) we have \( f(9, y) = 9y - y^2 - 9 \), a quadratic in \( y \) which on \([0, 5]\) attains its maximum at \( y = 9/2: \ f(9, 9/2) = 45/4 \) and its minimum at \( y = 0: \ f(9, 0) = -9 \).

On \( L_3 \) we have \( f(x, 5) = 5\sqrt{x} - x - 5 \), a function whose maximum on \([0, 9]\) is attained at \( x = 25/4: \ f(25/4, 5) = 5/4 \) and its minimum at \( x = 0: \ f(0, 5) = -5 \).
On $L_4$ we have $f(0, y) = -y^2 + 6y$, a quadratic in $y$ which on $[0, 5]$ attains its maximum at $y = 3$: $f(0, 3) = 9$ and its minimum at $y = 0$: $f(0, 0) = 0$.

Thus, the absolute maximum of $f$ on $D$ is $f(4, 4) = 12$ and the absolute minimum is $f(9, 0) = -9$.

(b) The first derivatives are: $f_x = 4x + 1$, $f_y = 2y$.

The only critical point is $(-1/4, 0)$, which is in $D$, and $f(-1/4, 0) = -17/8$.

On the boundary of $D$, the circle $x^2 + y^2 = 4$, we have $f(x, \pm \sqrt{4-x^2}) = x^2 + x + 2$, a quadratic in $x$ which in $[-2, 2]$ attains its minimum at $x = -1/2$, $f(-1/2, \pm \sqrt{15}/2) = 7/4$ and its maximum at $x = 2$, $f(2, 0) = 8$.

Thus, the absolute maximum of $f$ on $D$ is $f(2, 0) = 8$ and the absolute minimum is $f(-1/4, 0) = -17/8$. 