MATH 209 – Homework # 2

Solutions

Sec. 14.3. (pp 889-890 in [St])

Exc. 14.3.16
\[ f_x = 4x^3y^3 + 16xy, \quad f_y = 3x^4y^2 + 8x^2. \]

Exc. 14.3.24
\[ \frac{\partial w}{\partial u} = -e^v/(u + v^2)^2, \quad \frac{\partial w}{\partial v} = e^v/(u + v^2) - e^v/(u + v^2)^2 \cdot 2v = e^v(u + v^2 - 2v)/(u + v^2)^2. \]

Exc. 14.3.26
\[ f_x = \frac{\sqrt{t}}{1 + x^2t}, \quad f_t = \frac{1}{1 + x^2t}, \quad x = \frac{x}{2(1 + x^2t)^\frac{1}{2}}. \]

Exc. 14.3.31
\[ \frac{\partial w}{\partial x} = 1 + 2y + 3z, \quad \frac{\partial w}{\partial y} = 2x + 2y + 3z, \quad \frac{\partial w}{\partial y} = 3x + 2y + 3z. \]

Exc. 14.3.36
\[ f_x = \frac{y^2}{t + 2z}, \quad f_y = \frac{2xy}{t + 2z}, \quad f_z = -\frac{2xy^2}{(t + 2z)^2}, \quad f_t = -\frac{xy^2}{(t + 2z)^2}. \]

Exc. 14.3.45
To find \( \partial z/\partial x \), differentiate \( x^2 + y^2 + z^2 = 3xyz \) partially w.r.t. \( x \):
\[ 2x + 2z \frac{\partial z}{\partial x} = 3xz + 3xy \frac{\partial z}{\partial x} \implies (2z - 3xy) \frac{\partial z}{\partial x} = 3yz - 3x \implies \frac{\partial z}{\partial x} = \frac{2x - 3yz}{2z - 3xy}. \]

Similarly, differentiate \( x^2 + y^2 + z^2 = 3xyz \) partially w.r.t. \( y \):
\[ 2y + 2z \frac{\partial z}{\partial y} = 3xz + 3xy \frac{\partial z}{\partial y} \implies (2z - 3xy) \frac{\partial z}{\partial x} = 3xz - 3y \implies \frac{\partial z}{\partial y} = \frac{2y - 3xz}{2z - 3xy}. \]

(Note that \( x^2 + y^2 + z^2 = 3xyz \) does not change if \( x \) and \( y \) are interchanged. As a consequence, \( \partial z/\partial y \) could be obtained from \( \partial z/\partial x \) simply by interchanging \( x \) and \( y \), and vice versa.)

Exc. 14.3.48
To find \( \partial z/\partial x \), differentiate \( \sin(xyz) = x + 2y + 3z \) partially w.r.t. \( x \):
\[ \cos(xyz) \left( yz + xy \frac{\partial z}{\partial x} \right) = 1 + 3 \frac{\partial z}{\partial x} \implies (xy \cos(xyz) - 3) \frac{\partial z}{\partial x} = 1 - yz \cos(xyz) \implies \frac{\partial z}{\partial x} = \frac{1 - yz \cos(xyz)}{3 - xy \cos(xyz)}. \]

Analogously, differentiate \( \sin(xyz) = x + 2y + 3z \) partially w.r.t. \( y \):
\[ \cos(xyz) \left( xz + xy \frac{\partial z}{\partial y} \right) = 2 + 3 \frac{\partial z}{\partial y} \implies (xy \cos(xyz) - 3) \frac{\partial z}{\partial y} = 2 - xz \cos(xyz) \implies \frac{\partial z}{\partial y} = \frac{2 - xz \cos(xyz)}{3 - xy \cos(xyz)}. \]

Exc. 14.3.51
The first partial derivatives of \( f(x, y) = x^3y^5 + 2x^4y \) are
\[ f_x = 3x^2y^5 + 8x^3y, \quad f_y = 5x^3y^4 + 2x^4. \]
and hence the second partial derivatives are
\[ f_{xx} = 6xy^5 + 24x^2y, \quad f_{xy} = 15x^2y^4 + 8x^3 = f_{yx}, \quad f_{yy} = 20x^3y^3. \]

**Exc. 14.3.55**

The first partial derivatives of \( z = \arctan \frac{x+y}{1-xy} \) are
\[
\frac{\partial z}{\partial x} = \frac{1}{1 + (\frac{x+y}{1-xy})^2} \cdot \frac{1 - xy + (x+y)y}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2 + (x+y)^2},
\]
and, analogously,
\[
\frac{\partial z}{\partial y} = \frac{1}{1 + (\frac{x+y}{1-xy})^2} \cdot \frac{1 - xy + (x+y)x}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2 + (x+y)^2},
\]
(Note that \( \partial z/\partial x \) and \( \partial z/\partial y \) do not depend on \( y \) and \( x \), respectively.) Hence the second partial derivatives are
\[
\frac{\partial^2 z}{\partial x^2} = -\frac{2x}{(1 + x^2)^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = 0 = \frac{\partial^2 z}{\partial y \partial x}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{2y}{(1 + y^2)^2}.
\]

**Exc. 14.3.72**

(a) \( u_{xx} = 2, \ u_{yy} = 2 \), hence \( u_{xx} + u_{yy} = 4 \neq 0 \), and \( u = x^2 + y^2 \) is not a solution of Laplace’s equation.

(b) \( u_{xx} = 2, \ u_{yy} = -2 \), hence \( u_{xx} + u_{yy} = 0 \), showing that \( u = x^2 - y^2 \) is a solution.

(c) \( u_{xx} = 6x, \ u_{yy} = 6x \), hence \( u_{xx} + u_{yy} = 12x \neq 0 \), hence \( u = x^3 + 3xy^2 \) is not a solution.

(d) Since \( u = \ln \sqrt{x^2 + y^2} \) does not change if \( x \) and \( y \) are interchanged, the derivatives \( u_y \) and \( u_{yy} \) simply equal \( u_x \) and \( u_{xx} \), respectively, with \( x \) and \( y \) interchanged. Thus
\[
u_x = \frac{1}{\sqrt{x^2 + y^2}}, \quad \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{x^2 + y^2}, \quad v_y = \frac{y}{x^2 + y^2},
\]
as well as
\[
u_{xx} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - 2x^2}{(x^2 + y^2)^2}, \quad u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = -u_{xx},
\]
showing that \( u \) is a solution.

(e) \( u_{xx} = -\sin x \cosh y - \cos x \sinh y, \ u_{yy} = \sin x \cosh y + \cos x \sinh y = -u_{xx} \), hence \( u = \sin x \cosh y + \cos x \sinh y \) is a solution.

(f) \( u_{xx} = e^{-x} \cos y + e^{-y} \cos x, \ u_{yy} = -e^{-x} \cos y - e^{-y} \cos x = -u_{xx} \), hence \( u = e^{-x} \cos y - e^{-y} \cos x \) is a solution as well.

**Sec. 14.4. (pp 899-900 in [St])**

**Exc. 14.4.2**

From \( \partial z/\partial x = 6(x - 1) \) and \( \partial z/\partial y = 4(y + 3) \), it follows that \( \partial z/\partial x(2, -2) = 6, \partial z/\partial y(2, -2) = 4, \) and hence the tangent plane to \( z = 3(x - 1)^2 + 2(y + 3)^2 + 7 \) at \( (2, -2, 12) \) is given by \( z - 12 = 6(x - 2) + 4(y + 2) \) or, equivalently, \( z = 6x + 4y + 8 \).

**Exc. 14.4.3**

From \( \partial z/\partial x = \frac{1}{2} \sqrt{y/x} \) and \( \partial z/\partial y = \frac{1}{2} \sqrt{x/y} \), it follows that \( \partial z/\partial x(1, 1) = \frac{1}{2}, \partial z/\partial y(1, 1) = \frac{1}{2}, \) and hence the tangent plane to \( z = \sqrt{xy} \) at \( (1, 1, 1) \) is given by \( z - 1 = \frac{1}{2} (x - 1) + \frac{1}{2} (y - 1) \) or, equivalently, \( x + y = 2z \).
Exc. 14.4.5  
From $\partial z/\partial x = -y \sin(x-y)$ and $\partial z/\partial y = \cos(x-y)+y \sin(x-y)$, it follows that $\partial z/\partial x(2, 2) = 0$, $\partial z/\partial y(2, 2) = 1$, and hence the tangent plane to $z = y \cos(x-y)$ at $(2, 2, 2)$ is given by $z-2 = 0 \cdot (x-2) + 1 \cdot (y-2)$ or, equivalently, $y = z$.

Exc. 14.4.19  
The (first) partial derivatives of $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ are 

$$ f_x = -\frac{x}{\sqrt{20 - x^2 - 7y^2}}, \quad f_y = -\frac{7y}{\sqrt{20 - x^2 - 7y^2}}, $$

so that, specifically, $f(2, 1) = 3$, $f_x(2, 1) = -\frac{2}{3}$, $f_y(2, 1) = -\frac{7}{3}$. The linear approximation of $f$ near $(2, 1)$ is therefore given by 

$$ f(x, y) \approx 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1), $$

and in particular $f(1.95, 1.08) \approx 3 + \frac{2}{3} \cdot 0.05 - \frac{7}{3} \cdot 0.02 = \frac{447}{150} = 2.946$. (Note that the exact value of $f(1.95, 1.08)$ is $\sqrt{80327} \approx 2.834$, and hence the approximate value $\frac{447}{150}$ is a mere $0.44\%$ too large.)

Exc. 14.4.36  
The volume of the can is $V(R, H) = \pi R^2 H$, and hence $dV = 2\pi RH \, dR + \pi R^2 \, dH$. Letting $\delta$ denote the thickness of the metal in the sides, and thus $\Delta R = \delta$, $\Delta H = 4\delta$, an approximation for the necessary amount of metal is 

$$ \Delta V \approx 2\pi RH \Delta R + \pi R^2 \Delta H = 2\pi RH \delta + 4\pi R^2 \delta = 2\pi R(H + 2R)\delta. $$

With the specific values $H = 10\text{cm}$, $R = 2\text{cm}$, and $\delta = 0.05\text{cm}$, one finds $\Delta V \approx \frac{144}{10^3} \pi \approx 8.796\text{cm}^3$. (Note that the exact amount of metal would be 

$$ V(R, H) - V(R - \delta, H - 4\delta) = 2\pi R(H + 2R)\delta - \pi(H + 8R)\delta^2 + 4\pi \delta^3, $$

which, for the specific values of $H$, $R$, and $\delta$, evaluates to $\frac{5471}{10000} \pi \approx 8.594\text{cm}^3$.)

Additional problem (related to 14.4.46, p 900 in [St])

For every $(x, y) \neq (0, 0)$,

$$ f_x(x, y) = \frac{y(x^2 + y^2) - 2x^2y}{(x^2 + y^2)^2}, \quad f_y(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2}, $$

while

$$ f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0, \quad f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = 0. $$

Thus the partial derivatives $f_x$, $f_y$ exist for every $(x, y) \in \mathbb{R}^2$, and

$$ f_x(x, y) = \begin{cases} \frac{y(x^2 + y^2) - 2x^2y}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases} \quad f_y(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} $$

On the other hand, $f(h, h) = \frac{1}{2}$ for all $h \neq 0$, and so

$$ \lim_{h \to 0} f(h, h) = \frac{1}{2} \neq 0 = f(0, 0), $$

showing that $f$ is not continuous at $(0, 0)$. (You may want to compare this to the first-year calculus fact that $F = F(x)$ is continuous at $x_0$ whenever $F'(x_0)$ exists.)