Math 209 Solutions to Assignment 7

1. Use a triple integral to find volume of the solid bounded by the cylinder \( y = x^2 \) and the planes \( z = 0 \), \( z = 4 \) and \( y = 9 \).

**Solution.**

The solid region is \( E : -3 \leq x \leq 3, \ x^2 \leq y \leq 9, \ 0 \leq z \leq 4 \). Then

\[
V = \iiint_E dV = \int_{-3}^{3} \int_{x^2}^{9} \int_{0}^{4} dz \, dy \, dx = \int_{-3}^{3} \int_{x^2}^{9} \left[ \frac{4}{3} \right] dy \, dx
\]

\[
= 4(27 - 9 + 27 - 9) = 144.
\]

2. Use cylindrical coordinates to evaluate \( \iiint_E (x^3 + xy^2) \, dV \) where \( E \) is the solid in the first octant that lies beneath the paraboloid \( z = 1 - x^2 - y^2 \).

**Solution.**

The solid region \( E \) is: \( 0 \leq x \leq 1, \ 0 \leq y \leq \sqrt{1-x^2}, \ 0 \leq z \leq 1 - x^2 - y^2 \). So in cylindrical coordinates:

\[
E : 0 \leq \theta \leq \pi/2, \ 0 \leq r \leq 1, \ 0 \leq z \leq 1 - r^2.
\]

\[
\iiint_E (x^3 + xy^2) \, dV = \int_{0}^{\pi/2} \int_{0}^{1} \int_{0}^{1-r^2} \left[ r^3 \cos^3 \theta + (r \cos \theta)(r^2 \sin^2 \theta) \right] r \, dz \, dr \, d\theta
\]

\[
= \int_{0}^{\pi/2} \int_{0}^{1} \left[ r^4 \cos \theta \left( \sin^2 \theta \right) \right] dz \, dr \, d\theta
\]

\[
= \int_{0}^{\pi/2} \left[ \frac{r^5}{5} \left( \frac{1}{2} - \frac{1}{2} \right) \right] dr \, d\theta
\]

\[
= \frac{2}{35}.
\]

3. Use cylindrical coordinates to evaluate \( \iiint_E x^2 \, dV \) where \( E \) is the solid that lies within the cylinder \( x^2 + y^2 = 1 \), above the plane \( z = 0 \), and below the cone \( z^2 = 4x^2 + 4y^2 \).
Solution. \( E \): \( 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1, \ 0 \leq z \leq 2r. \)

\[
\iiint_E x^2 \, dV = \int_0^{2\pi} \int_0^1 \int_0^{2r} (r^2 \cos^2 \theta) r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 \left[ \int_0^{2r} (r^3 \cos^2 \theta) \, dr \right] d\theta = 2 \int_0^{2\pi} \int_0^1 2r^4 \cos^2 \theta \, dr \, d\theta = 2 \int_0^{2\pi} \cos^2 \theta \, d\theta \int_0^1 r^4 \, dr
\]

\[
= 2 \cdot \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta \left[ \frac{r^5}{5} \right]_0^1 = \frac{1}{5} \left[ \theta + \frac{1}{2} \sin 2\theta \right]^{2\pi}_0 = \frac{1}{5} (2\pi) = \frac{2\pi}{5}.
\]

4. Find the mass of a ball \( B \) given by \( x^2 + y^2 + z^2 \leq a^2 \) if the density at any point is proportional to its distance from the z-axis (use cylindrical coordinates).

Solution. Since the density is proportional to the distance from the z-axis, the density function is \( f(x, y, z) = k\sqrt{x^2 + y^2} \). The plane region \( D \) is a disk with radius \( a \). So \( E : 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq a, \ -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2} \). Then the mass \( m \) is

\[
m = \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} k\sqrt{r^2} a \, dz \, dr \, d\theta = 2k \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} r^2 \, dz \, dr \, d\theta
\]

\[
= 2k \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} \sqrt{a^2 - r^2} \, dr \, d\theta = 2k \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, dr \, d\theta
\]

\[
= 2k \int_0^{2\pi} \frac{a^2}{4} \sin^2 t (\cos t) a \cos t \, dt \quad \text{(let } r = a \sin t, \ dr = a \cos t \, dt) \]

\[
= 4k \pi a^4 \int_0^{\pi/2} \sin^2 t \cos^2 t \, dt = 4k a^4 \frac{1}{4} \int_0^{\pi/2} \sin^2 2t \, dt = ka^4 \pi \cdot \frac{1}{2} \int_0^{\pi/2} (1 - \cos 4t) \, dt
\]

\[
= \frac{ka^4 \pi}{2} \left[ t - \frac{1}{4} \sin 4t \right]_0^{\pi/2} = \frac{ka^4 \pi}{2} \left[ \frac{\pi}{2} \right] = \frac{ka^4 \pi^2}{4}.
\]

5. Find the gradient vector field of \( f(x, y) = \ln(x + 2y) \).

Solution. \( f_x(x, y) = \frac{1}{x + 2y} \) and \( f_y(x, y) = \frac{2}{x + 2y} \). So the gradient vector field is

\[
\nabla f(x, y) = \left\langle \frac{1}{x + 2y}, \frac{2}{x + 2y} \right\rangle = \frac{1}{x + 2y} \mathbf{i} + \frac{2}{x + 2y} \mathbf{j}.
\]

6. Find the gradient vector field of \( f(x, y, z) = x \cos \left( \frac{y}{z} \right) \).

Solution. \( f_x = \cos \left( \frac{y}{z} \right), \ f_y = -\frac{x}{z} \sin \left( \frac{y}{z} \right), \ f_z = \frac{xy}{z^2} \sin \left( \frac{y}{z} \right) \). Hence the gradient vector field of \( f \) is

\[
\nabla f(x, y) = \left\langle \cos \left( \frac{y}{z} \right), \ -\frac{x}{z} \sin \left( \frac{y}{z} \right), \ \frac{xy}{z^2} \sin \left( \frac{y}{z} \right) \right\rangle = \cos \left( \frac{y}{z} \right) \mathbf{i} - \frac{x}{z} \sin \left( \frac{y}{z} \right) \mathbf{j} + \frac{xy}{z^2} \sin \left( \frac{y}{z} \right) \mathbf{k}.
\]
7. Evaluate the line integral \( \int_C \sin x \, dx + \cos y \, dy \) where \( C \) consists of the top half of the circle \( x^2 + y^2 = 1 \) from \((1, 0)\) to \((-1, 0)\) and the line segment from \((-1, 0)\) to \((-2, 3)\).

**Solution.**

Let \( C_1 \) be the path on the circle from \((1, 0)\) to \((-1, 0)\) and \( C_2 \) be the line segment from \((-1, 0)\) to \((-2, 3)\). Then

\[
C_1: \begin{cases} 
    x = \cos t, \\
    y = \sin t, \\
0 \leq t \leq \pi;
\end{cases}
\]

\[
C_2: \begin{cases} 
    x = (1 - t)(-1) + t(-2) = -1 - t, \\
    y = (1 - t)(0) + t(3) = 3t, \\
0 \leq t \leq 1.
\end{cases}
\]

Then

\[
\int_C \sin x \, dx + \cos y \, dy = \int_{C_1} \sin x \, dx + \cos y \, dy + \int_{C_2} \sin x \, dx + \cos y \, dy
\]

\[
= \int_0^\pi \sin(u) du - \int_0^\pi \cos(v) dv + \int_0^1 \sin(-(1 + t))(\cos(t)) dt + \cos(3t)(3) dt
\]

\[
= \left[ -\cos(u) + \sin(v) \right]_0^\pi + \left[ -\cos(1 + t) + 3 \cdot \frac{1}{3} \sin(3t) \right]_0^1
\]

\[
= -\cos(\pi) + \sin(\pi) + \cos(0) - \sin(0) - \cos 2 + \sin 3 + \cos 1 - \sin 0
\]

\[
= -\cos(-1) + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3.
\]

8. Evaluate the line integral \( \int_C x^2 z \, ds \) where \( C \) is the line segment from \((0, 6, -1)\) to \((4, 1, 5)\).

**Solution.** \( C : \ r(t) = (1 - t)(0, 6, -1) + t(4, 1, 5) = (4t, 6 - 5t, -1 + 6t) \) where \( 0 \leq t \leq 1 \). Therefore,

\[
C: \ x = 4t, \quad y = 6 - 5t, \quad z = -1 + 6t, \quad 0 \leq t \leq 1.
\]

Then

\[
\int_C x^2 z \, ds = \int_C x^2(t) z(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]

\[
= \int_0^1 (4t)^2(6t - 1) \sqrt{4^2 + (-5)^2 + 6^2} \, dt = 16\sqrt{77} \int_0^1 t^2(6t - 1) \, dt = 16\sqrt{77} \int_0^1 (6t^3 - t^2) \, dt
\]

\[
= 16\sqrt{77} \left[ \frac{6t^4}{4} - \frac{t^3}{3} \right]_0^1 = 16\sqrt{77} \left( \frac{3}{2} - \frac{1}{3} \right) = 16\sqrt{77} \left( \frac{7}{6} \right) = \frac{56}{3} \sqrt{77}.
\]
9. Evaluate the line integral \( \int_C x^2 \, dx + y^2 \, dy + z^2 \, dz \) where \( C \) consists of the line segments from \((0, 0, 0)\) to \((1, 2, -1)\) and from \((1, 2, -1)\) to \((3, 2, 0)\).

**Solution.** Let \( C_1 \) and \( C_2 \) be the line segments from \((0, 0, 0)\) to \((1, 2, -1)\) and from \((1, 2, -1)\), resp. Then for \(0 \leq t \leq 1\)

\[
C_1 : \quad r = (1-t)(0, 0, 0) + t(1, 2, -1) = (t, 2t, -1)
\]

\[
C_2 : \quad r = (1-t)(1, 2, -1) + t(3, 2, 0) = (1 + 2t, 2, -1 + t).
\]

Thus,

\[
\begin{align*}
\int_C x^2 \, dx + y^2 \, dy + z^2 \, dz &= \int_0^1 \left( (1 + 2t)^2 (2 \, dt) + 4 \cdot 0 \, dt + (t - 1)^2 \, dt + \int_0^1 t^2 \, dt + 4t^2(2) \, dt + t^2(-1) \, dt \right) \\
&= \int_0^1 (2(1 + 4t^2)) + t^2 - 2t + 1) \, dt + \int_0^1 8t^2 \, dt \\
&= \int_0^1 (9t^2 + 6t + 3) \, dt + \int_0^1 8t^2 \, dt = \frac{9t^3}{3} + \frac{6t^2}{2} + 3t \left|_0^1 \right. + \frac{8t^3}{3} \left|_0^1 \\
&= 3 + 3 + \frac{8}{3} = \frac{35}{3}.
\end{align*}
\]

10. If \( f(x, y) = x^2 - y^2 \), find \( \int_C f(x, y) \, ds \) where \( C \) consists of the edges of a triangle with vertices at \((0, 0)\), \((2, 1)\) and \((1, 2)\).

**Solution.** Let \( C_1, C_2 \) and \( C_3 \) be the line segments from \((0, 0)\) to \((2, 1)\), from \((2, 1)\) to \((1, 2)\) and from \((1, 2)\) to \((0, 0)\), respectively. Then \( C = C_1 + C_2 + C_3 \) where the parametric equations of the line segments are:

\[
C_1 : \begin{aligned}
x &= 0 \cdot (1 - t) + 2t = 2t, \\
y &= 0 \cdot (1 - t) + t = t,
\end{aligned} \quad 0 \leq t \leq 1;
\]

\[
C_2 : \begin{aligned}
x &= 2(1 - t) + t = 2 - t, \\
y &= 1(1 - t) + 2t = t + 1,
\end{aligned} \quad 0 \leq t \leq 1;
\]

\[
C_3 : \begin{aligned}
x &= 1(1 - t) + 0 \cdot t = 1 - t, \\
y &= 2(1 - t) + 0 \cdot t = 2 - 2t,
\end{aligned} \quad 0 \leq t \leq 1.
\]

Therefore,

\[
\begin{align*}
\int_C (x^2 - y^2) \, ds &= \int_{C_1} (x^2 - y^2) \, ds + \int_{C_2} (x^2 - y^2) \, ds + \int_{C_3} (x^2 - y^2) \, ds \\
&= \int_0^1 (4t^2 - t^2)\sqrt{5} \, dt + \int_0^1 [(2 - t)^2 - (t + 1)^2]\sqrt{2} \, dt + \int_0^1 [(1 - t)^2 - 4(1 - t)^2]\sqrt{5} \, dt \\
&= 3\sqrt{5} \left[ \int_0^1 t^2 \, dt + \sqrt{2} \int_0^1 (3 - 6t) \, dt + \sqrt{5} \int_0^1 (-3 + 6t - 3t^2) \, dt \right] \\
&= 3\sqrt{5} \left[ \left. \frac{3t^3}{3} \right|_0^1 + \sqrt{2} \left. (3t - 3t^2) \right|_0^1 + \sqrt{5} \left. (-3t + 3t^2 - t^3) \right|_0^1 = \sqrt{5} + \sqrt{2}(0) + \sqrt{5}(-1) = 0. \right.
\end{align*}
\]